Wave Numbers - A Number System without Negative Numbers

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Abstract

This paper introduces the Wave Number system, a novel mathematical framework that operates without the inclusion of negative numbers, challenging the traditional necessity of negative and imaginary numbers in algebraic structures. The motivation for this system stems from the historical scepticism surrounding negative numbers and their eventual acceptance, contrasting with the effective functioning of the double-entry bookkeeping system without them. The fundamental equation of the Wave Number system is $\mathbf{1}\hat{} + \mathbf{1}^{v} = \mathbf{0}$, and it is developed within the context of $\mathbb{R}\mathbf{3}$, adhering to the algebraic permanence principle. Key concepts introduced include Opposite Values, a redefined concept of "less than," and the new operation of rotication. The paper provides a comprehensive set of axioms and a detailed description of operations such as addition, rotation, rotication, multiplication, dot-product, cross-product, and division. Computational methods and formulae are outlined, with all calculations verified against an online Wave Number calculator implemented in Python and hosted on Amazon Web Services. The Wave Number system's structure is demonstrated to be more computationally efficient than quaternions, particularly in rotation operations, making it a viable alternative for applications requiring such computations. The paper concludes by affirming the feasibility and efficiency of the Wave Number system as a complete and practical number system.

Keywords: Wave Number, Opposite Value, Rotation, Quaternion

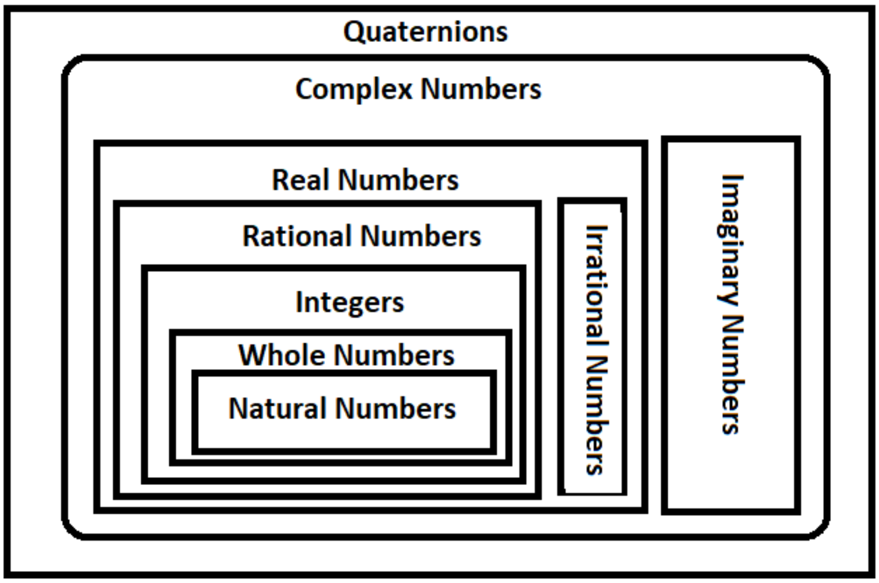

Fig. 1 Today's Number Systems

1 Introduction

The initial use of natural numbers was for counting, with the earliest written evidence found in Babylonian tablets dating from 1830 to 1501 BCE.[1] These tablets employed a sexagesimal number system with a base of 60, utilizing a positional setup to represent powers of 60. Early tablets lacked a symbol for zero; however, by 400 BCE, Babylonians used two oblique wedges to indicate a missing power of 60, similar to the modern use of zero. The sexagesimal system facilitated addition, subtraction, multiplication, and division of whole numbers. Natural numbers serve as the innermost building block of the number systems we use today, as shown in Figure 1.

The inclusion of division led to the introduction of fractions and, consequently, rational numbers. Around 400 BCE, Plato described the lesson of Theodorus, in which a student learns about the "incommensurability" or irrationality of the square roots of 3, 5, and 6.[1] By approximately 500 BCE, Euclid had identified 23 classes of irrational numbers.

Whole numbers extend the natural numbers by including zero. Brahmagupta first formalized zero in his treatise Brahmasphutasiddhanta in 628 CE, where he also described negative whole numbers.[2] Consequently, by this time, mathematicians had defined the elements of integers, as integers encompass all whole numbers and negative whole numbers. Similarly, since real numbers include negatives, the elements of real numbers had also been defined by this time.

In Ars Magna (1545), Cardano published a formula for cubic equations that necessitated the use of square roots of negative numbers.[1] Two decades later, Bombelli introduced the concept that $\sqrt{-1} * \sqrt{-1} = -1$, thereby formalizing the concept of complex numbers.[1]

In Chapter XII of Euler's Elements of Algebra (1700), he referred to the square roots of negative numbers as "imaginary quantities" because they exist merely in the imagination.[3] He was the first to refer to the $\sqrt{-1}$ as i in a memoir, De Formulis Differentialibus Angularibus, where he wrote "In the following, I shall denote the expression $\sqrt{-1}$ by i so that $ii = -1$."[4] Thus, the two elements of complex numbers, real numbers and imaginary numbers, were defined.

William Rowan Hamilton (1805-1865) posited that any complex number can be represented by a pair (a, b) of real numbers on the Cartesian plane.[1] In his efforts to extend complex numbers to higher dimensions, Hamilton introduced quaternions. To formalize this new algebra, he added two new imaginary units, j and k, alongside the imaginary unit i. Similar to how i represents a 90-degree rotation around the x-axis, j represents a 90-degree rotation around the y-axis, and k represents a 90-degree rotation around the z-axis. This framework culminated in his renowned equation $i^2 = j^2 = k^2 = ijk = -1$.

Complex numbers emerged as a natural extension of the concept of negative numbers, a development that was not immediately embraced by the mathematical community. Initially, negative numbers were met with scepticism and were not universally accepted.

Michael Stifel (1487 -1567) was one of the early mathematicians to formalize the arithmetic rules for negative numbers,[1] though he referred to them as "numeri absurdi".[5] Later, John Napier (1550-1617), the inventor of logarithms, described negative numbers as "defectivi".[5] This historical resistance to negative numbers mirrors contemporary challenges faced by students. It is well-documented in the field of mathematics education that children often struggle with the concept of negative numbers when they first encounter them.[2]

Fra Luca Bartolomeo de Pacioli is celebrated as "The Father of Accounting and Bookkeeping" for his foundational work on double-entry bookkeeping, which he first detailed in his 1494 algebraic treatise Summa de arithmetica, geometria, proportioni et proportionalita.[6] In this book, Pacioli introduces a system where debits serve to increase expense accounts and credits serve to decrease them, while credits increase revenue accounts and debits reduce them. Remarkably, this bookkeeping system operates effectively without the explicit use of negative numbers.

This observation prompts a deeper inquiry into whether a number system can be constructed and maintained without the inclusion of negative numbers.

This paper introduces the Wave Number system, as depicted in Figure 2, along with its accompanying on-line calculator at www.wavenumbers.com/calculator. The Wave Number system is distinctive in that it operates without the use of negative numbers. Central to this system is the fundamental equation:

$$1^{\hat{}} + 1^{v} = 0$$

The Wave Number system facilitates an algebraic structure in $\mathbb{R}3$ that operates without the use of negative numbers, thereby obviating the necessity for imaginary numbers, as all quantities remain within the realm of real numbers. This system identifies rotation as a fundamental mathematical operation and introduces a novel operation termed "rotication". Among various notable findings, the analysis

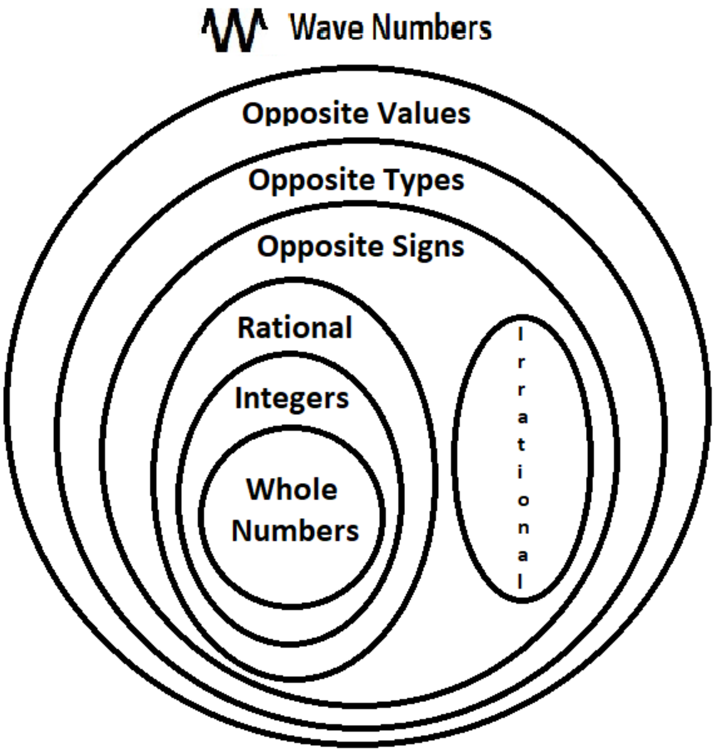

Fig. 2 Wave Number System

demonstrates that the Wave Number system offers superior computational efficiency compared to quaternions for rotation operations.

The paper is organized as follows: Section 2 reviews related work, including axiom design and quaternions. Section 3 describes the methods used, focusing on the design approach for the system and the development of the online calculator. Section 4 outlines the fundamental principles of the Wave Number system. Section 5 details the axioms underlying the system. Section 6 elaborates on the main mathematical operations within the Wave Number system, providing the relevant formulae and examples for each operation. Section 7 evaluates the results and discusses some intriguing properties of Wave Numbers. Section 8 presents the main conclusions and offers suggestions for future development of the Wave Number system.

2 Related Work

2.1 Axiom Creation

The Axioms for Real Numbers from the Department of Mathematics at the University of Washington are presented in a meticulously structured format for defining axioms and primitives.[7]

Similarly, the development of the Wave Number system's axioms and primitives employs this well-considered methodology.

2.2 Quaternions

The quaternion number system facilitates the performance of rotations in three dimensions. In this paper, quaternions are utilized to verify the accurate calculation of $\mathbb{R}3$ rotations within the Wave Number system.

The coordinates of a point in quaternion notation are expressed as

$$(a + bi + cj + dk)$$

where a is a scalar, and the terms bi, cj and dk represent rotations around the x, y and z-axes, respectively. Quaternions extend complex numbers and are based on a four-dimensional sphere used to represent three-dimensional rotations. Despite their complexity and challenging visualization, quaternions effectively support 3D multiplication and rotation. [8].

To convert a 3D point into a quaternion, assign it a scalar of 0 and use its Cartesian coordinates as the coefficients for i, j and k.

Quaternion multiplications can be performed either singly or in a chain. However, the outcome is a quaternion representing four dimensions, rather than a set of 3D Cartesian coordinates. To convert this four-dimensional result back into three-dimensional coordinates, a method known as the quaternion sandwich is employed. This involves performing the original quaternion multiplications in reverse order using the conjugates of the quaternions.[9]

The quaternion sandwich process, for a chain of two rotations, starts by multiplying a point P first by quaternion $q1$ and then by quaternion $q2$. Subsequently, the product is multiplied by the conjugates of $q1$, $(q1')$ and $q2$, $(q2')$. This process is represented by the formula:

$$q2 * (q1 * P * q1') * q2' = q3 * P * q3$$

where the conjugate of a quaternion $(a + bi + cj + dk)$ is defined as:

$$(a + bi + cj + dk)' = (a - bi - cj - dk)$$

3 Methods Used

3.1 System Design

In designing the Wave Number system, an effort has been made to adhere to the algebraic permanence principle.[2]

> "The principle states that if we want to extend the definition of a basic algebraic operation beyond its original domain (e.g., from the set $\mathbb{N}$ of natural numbers to the set $\mathbb{Z}$ of integers), then among all logically possible (non-contradictory) extensions the one to be chosen is that which best preserves the rules of calculation."

3.2 Calculator Development

The calculator (www.wavenumbers.com/calculator) is implemented in Python and is encapsulated to run on an Amazon Web Server. It communicates with the website through an Application Programming Interface (API).

The calculator supports operations such as addition, multiplication, division, root extraction, dot product, cross product, rotation, rotication, memory addition, memory flip, and memory clearing.

Simultaneous equations are employed for solving division and determining cubic roots. The Python package sympy.nonlinsolve is used to solve the equations related to division.[10] For solving cubic root equations, the scipy.optimize.fsolve package is utilized.[11]

4 Fundamental Principles

4.1 Introduction

This paper defines the fundamental mathematical principles of the Wave Number system. The Wave Number principles introduce the concept of $\hat{}$ Opposite Values (referred to as "Hat" Opposites) and ^{v} Opposite Values (referred to as "Vee" Opposites). All Opposite Values in the Wave Number system originate from 0, with no positive or negative values. It is important to note that the term "Opposite Value" is used instead of "Opposite Number" because an Opposite Value encompasses more information than a traditional number in classical mathematics.

The symbol for the Wave Number system is . This symbol represents the peaks and troughs of waves and consists of alternating Hats and Vees.

4.2 Opposite Values

Each Opposite Value (except zero) in the Wave Number system consists of three components:

1. Counter: This component provides the magnitude of the Opposite Value.
2. Opposite Type: This component indicates the axis on which the Opposite Value is located. For example, $1\hat{}$ is on the x-axis, $1i\hat{}$ is on the y-axis, and $1j\hat{}$ is on the z-axis. Note that the x-axis does not use a letter designation.
3. Opposite Sign: This component, either $\hat{}$ (called "Hat") or ^{v} (called "Vee"), determines which side of 0 on the Wave Number Line the Opposite Value is located.

"Hat" Opposite Values cancel out "Vee" Opposite Values with the same Counter, analogous to how two waves interfere and cancel each other out when they meet, provided that the waves are equal in magnitude and totally out of phase.

Opposite Values possess a value, whereas Counters do not, as they represent a count or magnitude. While calculations utilize Counters, it is not possible to have a result that is solely a Counter.

The absolute value of an Opposite Value is its magnitude, represented by the Counter. Consequently, Opposite Values with the same Counter but different signs or axes have identical magnitudes.

For example, $\hat{1}$ is the opposite of 1^v, and 1^v is the opposite of $\hat{1}$. The only difference between 1^v and $\hat{1}$ is their opposition, causing them to cancel each other out when added. Thus, $\hat{1}$ is the additive inverse of 1^v, and vice versa. This principle forms the foundation of Wave Numbers:

$$\hat{1} + 1^v = 0$$

4.3 Wave Number Lines

The next component of the Wave Number Principles is the Wave Number lines. Each dimension contributes its own unique Wave Number line.

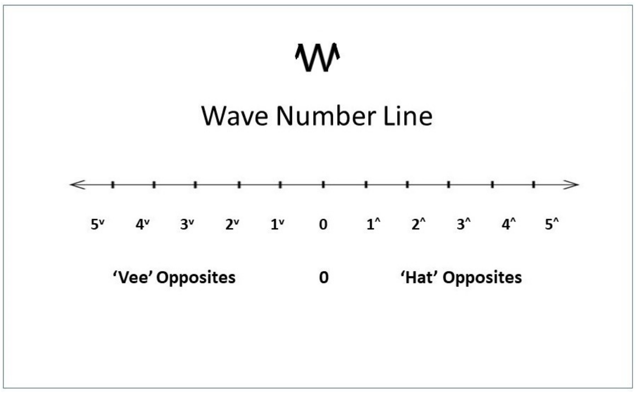

Fig. 3 R1: Wave Number Line

The notation R followed by the number of dimensions in the space represents the various Wave Number coordinate systems.

For example, the Wave Number line for one dimension is denoted as R1, which is equivalent to the classical number line as in Figure 3.

The Wave Number lines for two dimensions, denoted as R2, correspond to the classical x-y plane as in Figure 4.

Similarly, the x, y, and z axes are utilized in the Wave Number lines for the three-dimensional system, denoted as R3, analogous to the classical three-dimensional coordinate system.

Opposite Values on the x-axis are in the form $\hat{1}$ and 1^v, on the y-axis in the form $1\hat{i}$ and $1i^v$, and on the z-axis in the form $1\hat{j}$ and $1j^v$.

Typically, the Opposite Values $1\hat{i}$, $1i^v$, $1\hat{j}$, and $1j^v$ are written as $\hat{i}$, i^v, $\hat{j}$, and j^v, with the Counter 1 implied. These Opposite Values, along with $\hat{1}$ and 1^v, are referred to as unitaries.

According to the algebraic permanence principle, each dimension adheres to the same mathematical rules. Higher dimensions are incorporated using the same rules.

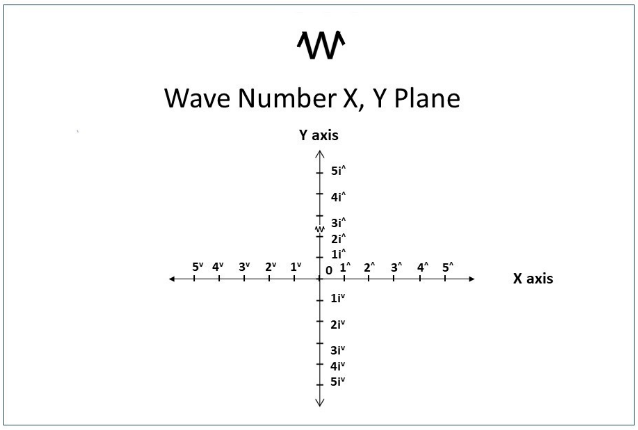

Fig. 4 R2: Wave Number X, Y Plane

Each axis contains 2 unitaries. A unitary Opposite Value has a magnitude of 1. In R1, the unitaries are $1^{\wedge}$ and 1^{v}. In R2, they are $1^{\wedge}$, 1^{v}, $i^{\wedge}$, and i^{v}, and so on.

4.4 Coordinate Systems

The Wave Number system utilizes Cartesian coordinates represented as Opposite Values. This allows for the addition of coordinates and multiplication by scalars. In classical mathematics, Cartesian coordinates cannot be multiplied together without introducing complex numbers.

In the Wave Number system, there is no distinction between a complex number system and a two-dimensional system for real values. All numbers are treated as real, permitting the direct multiplication of Cartesian coordinates.

4.5 Zero

In classical mathematics, zero (0) is not regarded as a number in the usual sense, yet it plays an extremely useful and fundamental role in various mathematical structures. The Wave Number system treats zero similarly to classical mathematics, considering it a special Opposite Value. Zero serves as a placeholder indicating the absence of a quantity, aiding in the representation of numbers. It marks the point on an axis where Opposite Values of different Opposite Signs converge, effectively representing the origin and denoting a state of lacking of value or emptiness.

The primary difference in the use of zero in the Wave Number system compared to classical mathematics is that no other Opposite Value is less than or equal to zero. Consequently, the concept of "less than" is redefined based on the magnitudes represented by the Counters of Opposite Values. Thus, all other Opposite Values are considered greater than zero.

4.6 Hats and Vees

A Wave Number can have either a $\hat{}$ or a ^{v} orientation. Visualize a $\hat{}$ Opposite Value as directed upward or to the right in R2, and a ^{v} Opposite Value as directed downward or to the left. This convention is arbitrary and could be reversed. The key concept is that they are opposites and can cancel each other out.

4.7 No Negativity

A fundamental principle of the Wave Numbers system is the absence of the concept of negativity. Similar to the dichotomy of good and evil, the concept of positivity becomes irrelevant without its negative counterpart. Wave Numbers are represented by $\hat{}$ and ^{v} Opposite Values that can cancel each other out.

As a result, the subtraction operation is unnecessary. While the addition sign is retained for clarity, it is not essential. It is assumed that all Opposite Values in an equation are added together. Historically, Diophantus did not find it necessary to use an addition sign.[1] In the Wave Numbers system, the addition sign is used solely as a placeholder between Opposite Values. Instead of interpreting numbers as being added together, they can be seen as interfering in the same way waves do when they meet.

However, it is still necessary at times to determine the difference between two Opposite Values. This is achieved by changing the Opposite Sign of one of the Opposite Values and then adding the two Opposite Values together. This process is known as flipping the Opposite Sign.

4.8 Counters

Counters can be visualized as existing on a vertical Counter line that starts at 0 and increases as their magnitudes grow larger. A Counter does not possess a positive or negative value; it reflects size, analogous to height, as it indicates the magnitude of an Opposite Value.

Special cases of Counters include constants such as π (pi), e (Euler's number), and ϕ (the golden ratio). These constants can be combined with Opposite Types and Signs ($\hat{}$, ^{v}) to form Opposite Values.

4.9 Rotation is a Fundamental Operation

Many differences between the Wave Number system and the classical system arise directly from the fundamental equation $1\hat{} + 1^v = 0$. Another principle of the Wave Number system is the treatment of rotation as a fundamental operation, on par with addition and multiplication. Rotation, denoted by the symbol $\circlearrowleft$, moves a point in a unique manner similar to how addition and multiplication operate.

Incorporating Opposite Values and rotation into the Wave Number system leads to the treatment of all Opposite Values in all dimensions as Real Opposite Values.

The same rotation rules apply to numbers in any dimension, although this can yield surprising results for operations such as multiplication and division.

4.10 R3 Algebra

The mathematical principles of the Wave Numbers system support an algebra in R3. Later, it will be demonstrated that the results of operations on Wave Numbers in R3. correspond to the results of similar operations on vectors in quaternions.

4.11 Operations

4.11.1 Introduction

Wave Numbers require a re-evaluation of general mathematical operations such as addition, subtraction, rotation, multiplication, and division. Operators, operations, and operands need new definitions within this system.

In Wave Number mathematics, the key distinction between $1^{\wedge}$ and 1^{v} is that they cancel each other out when they combine.

Subtraction is not used in Wave Numbers. Calculating the difference between two Opposite Values involves adding them after changing the Opposite Sign of one value using the flip operation $(^{-})$. The flip operation, described later, essentially rotates an Opposite Value by π. This is not a subtraction sign but an operation that changes the Opposite Sign $(^{\wedge}, ^{v})$ of an Opposite Value.

Thus, the cancellation effect typically provided by subtraction is achieved in Wave Numbers through addition with the Opposite Sign flipped.

To find the difference involving a variable with an unknown Opposite Sign, precede the variable with a flip sign $(^{-})$.

The sub-section below on rotation demonstrates that 1^{v} is equivalent to $1^{\wedge}$ rotated by an angle of π in either a clockwise or counterclockwise direction. Similarly, $1^{\wedge}$ is equivalent to 1^{v} rotated by an angle of π in either direction.

The Wave Number system introduces some new operations. The first is rotication, which multiplies the result of an initial rotation around an axis by the magnitude of the axis of rotation.

Additionally, new operations include rotvision and roticvision. Analogous to how division is the inverse of multiplication, rotvision is the inverse of rotation, and roticvision is the inverse of rotication.

4.11.2 Operators, Operations and Operands

A mathematical operation consists of an Operator, an Operation, and an Operand. For example, consider the expression $1^{\wedge} * 1^{v} = 1^{v}$. In this context, $1^{\wedge}$ is the operator, * denotes the operation (multiplication), and 1^{v} is the operand. The Opposite Type and Sign of the result depend on the multiplication table described later.

In Wave Number multiplication, the Operator performs the Operation on the Operand. Specifically, the Operator is the first term in the expression, and the Operand is the second term.

In the context of division, the Operand is the numerator, and the Operator functions as the divisor. While the Operator may be a Counter, the Operand must always be an Opposite Value in both multiplication and division.

For instance, consider the expression $1^{\hat{}}/1^v = 1^v$. Here, $1^{\hat{}}$ is the Operand, / represents the Operation of division, and 1^v serves as the Operator. The sign of the result is determined by the division table, which will be explained in subsequent sections.

In the context of roots, the Operator specifies the degree of the root, while the Operand represents the Opposite Value for which the root is being calculated.

For rotation, the Operator consists of two components: a radian or degree value and an axis of rotation. The symbol $\circlearrowleft$ denotes the rotation operation. The Operand represents the point being rotated, with an Opposite Value used to describe its position. For example, $\pi^{\hat{}}$ indicates a counterclockwise rotation of 180^o around the x-axis, while π^v represents a clockwise rotation of 180^{vo} around the y-axis.

Rotication follows the same format as rotation where the symbol for the Operation is R$\circlearrowleft$.

In the addition operation, add the Operator to the Operand. Both must be Opposite Values.

4.11.3 Addition

In the Wave Number system, only Opposite Values of the same Opposite Type can be consolidated through addition. This process involves adjusting the coordinates of a point on the Wave Number axis according to the types of Opposite Values present.

For example, in R2, the addition of Opposite Values is performed as follows:

$$(1^{\hat{}} + 2i^v) + (2^{\hat{}} + 3i^v) = (3^{\hat{}} + 5i^v)$$

4.11.4 Multiplication

In the Wave Numbers system, multiplication takes into account the $^{\hat{}}$ and ^{v} signs. When the Operator, which is the first term in a multiplication, is $^{\hat{}}$ then it denotes a 90^o turn counterclockwise during the operation. If there is a ^{v} in the first term in the multiplication then it represents a 90^{vo} turn clockwise. So, $i^{\hat{}} * 1^{\hat{}}$, in R2, results in the value of $i^{\hat{}}$ which is on the y-axis.

Multiplication by a unitary is the equivalent of rotation around the unitary's axis by 90^o. In this way unitaries link rotation and multiplication. The section on rotation explains this fully.

4.11.5 Rotation

Introducing the concept of rotation in the Wave Number system gives rise to the notions of R2 and the x-y plane. Consider the expression $a \circlearrowleft b$, where a is the Operator, b is the Operand, and $\circlearrowleft$ denotes the rotation Operation.

For example, the expression $\frac{\pi^{\hat{}}}{2} \circlearrowleft 1^{\hat{}} = i^{\hat{}}$ in R2 represents a counterclockwise rotation of 90^o around an imaginary z-axis. This operation moves the point $1^{\hat{}}$ on the x-axis to $i^{\hat{}}$ on the y-axis. Mathematically, this rotation transforms the point from $(1^{\hat{}}, 0)$ to $(0, i^{\hat{}})$, corresponding to $i^{\hat{}}$ algebraically.

Similarly, the expression $\frac{\pi^v}{2} \circlearrowleft 1^{\hat{}} = i^v$ represents a clockwise rotation of 90^{vo} around an imaginary z-axis. The operation moves the point $1^{\hat{}}$ on the x-axis to i^v on the y-axis, transforming the point from $(1^{\hat{}}, 0)$ to $(0, i^v)$, which corresponds to i^v algebraically.

Points between the axes are determined as the sum of values on the respective axes. For instance, in R2, the point $(\frac{\sqrt{2}}{2}\hat{} + \frac{\sqrt{2}}{2}\hat{i})$ represents the Cartesian coordinates $(\frac{\sqrt{2}}{2}\hat{}, \frac{\sqrt{2}}{2}\hat{i})$. This point results from a $45\hat{}^{o}$ counterclockwise rotation (or $\frac{\pi}{4}\hat{}$ radians) of the point $(\hat{1}, 0)$ to $(\frac{\sqrt{2}}{2}\hat{}, \frac{\sqrt{2}}{2}\hat{i})$ around the origin $(0, 0)$.

Trigonometric functions such as cosine, sine etc. are Counters as they are ratios, and so magnitudes, and are calculated based on the Counters of a point's coordinates, as in classical math. When calculating the angle of rotation of points on the x-y plane, the quadrant needs to be taken into account. For example, take the point at $(\hat{1}, \sqrt{3}\hat{})$, its radius is $\sqrt{|1^2| + |\sqrt{3}|^2} = 2$. Based on the Counters of its coordinates, its cosine is $1/2 = 0.5$ and its sine is $\sqrt{3}/2 = 0.866$. Its angle is $60\hat{}^{o}$.
Now take the point at $(1^v, \sqrt{3}^v)$, its radius is $\sqrt{|1^2| + |\sqrt{3}|^2} = 2$. Its cosine is $^-1/2 =^- 0.5$ and its sine is $^-\sqrt{3}/2 =^- 0.866$. Its angle is $240\hat{}^{o}$.

Note that the ^{v} coordinates are reflected in the equations by the applying flip signs $(^-)$ to the Counters.

4.11.6 Counters

An operation can involve multiple Counters that interact with each other. For instance, in the expression $6 * 4 * \hat{2}$, the Counters 6 and 4 are multiplied together.

Counters with flip signs can cancel each other out, as demonstrated by $^-6 *^- 4 * \hat{2} = 6 * 4 * \hat{2}$. In certain contexts, such as exponentiation and logarithms, Counters may also be added together.

4.11.7 Other

The double-dagger symbol (‡) is used in certain contexts to denote the Opposite Type and Sign of a variable.

For example, with inverse exponents as in $x^{-n} = 1 \ddagger x^{\ddagger x} / x^n$ in R3. Here, the double-daggers indicate that the numerator has the same Opposite Type and Sign as x. The $\ddagger x$ represents the Opposite Type of x, and the superscript $^{\ddagger x}$ denotes the Opposite Sign of x.

The symbol ? represents an unknown Opposite Type. The $^?$ superscript signifies an unknown Opposite Sign.

A preceding $^?$ indicates an unknown flip sign, which may or may not be present.

5 Axioms

5.1 Introduction

The Axioms begin with the definition of a set of primitives that form the foundation of the Wave Number system. Following this, definitions, axioms, and theorems are introduced. Finally, the use of ratios and the property of order within the Wave Number system are examined.

5.2 Primitives

This section defines the primitives that constitute the foundational concepts of the Wave Number system. These primitives are not defined in terms of other concepts.

Axioms for Real Numbers [7] define mathematical primitives as follows:

> "To avoid circularity, we cannot give every term a rigorous mathematical definition; we must accept some terms as undefined. For this context, we will consider the following fundamental notions as primitive undefined terms. You already understand what these terms mean; however, the only facts about them that can be used in proofs are those expressed in the following axioms (and any theorems that can be derived from these axioms)."

5.2.1 Opposite Values

Intuitively, an Opposite Value stands for a point on the number line. It represents a distance to the right ($\hat{}$) or left (^{v}) of the origin. This includes any finite or infinite decimal value. Opposite Values encompass integers, fractions, and irrational Opposite Values such as $\sqrt{2}\hat{}$, $\sqrt{2}^v$, $\pi\hat{}$, π^v, $e\hat{}$ and e^v.

5.2.2 Integer

An integer is a whole Opposite Value ($?\hat{}$, $?^v$ or zero).

5.2.3 Zero

The Opposite Value zero is denoted by 0.

5.2.4 One

One (1) represents the magnitude of the Opposite Value at one unit of distance from the origin.

5.2.5 Addition

Adding two Opposite Values α and β is represented by $\alpha + \beta$ and is named the sum of α and β. Notably, the sum of $\alpha\hat{}$ and α^v is 0.

5.2.6 Multiplication

Multiplying two Opposite Values α and β is represented by $\alpha * \beta$ and is named the product of α and β.

5.2.7 Less Than

The expression that α is less than β, represented by $\alpha < \beta$, indicates that the magnitude of α is smaller than the magnitude of β.

5.3 Definitions

5.3.1 Opposite Values

The system utilizes two types of Opposite Values: $\hat{}$ (called "Hat") and ^{v} (called "Vee"). These are constructed using Counters, Opposite Types, and Opposite Signs.

The Wave Number system uses the term "Opposite Values" instead of "Opposite numbers" because an Opposite Value encompasses more information than a pure number in classical mathematics.

The foundational equation of the Wave Number system reflects the opposing nature of Opposite Values and is expressed as follows:

$$\hat{1} + 1^v = 0$$

5.3.2 Counters

An Opposite Value is composed of three parts. The first part is the Counter, which determines the magnitude of the Opposite Value. The absolute value of an Opposite Value is its Counter, denoted as $|\alpha|$ where α is the Opposite Value.

5.3.3 Opposite Types

The second component of an Opposite Value is the Opposite Type, which indicates the axis on which the Opposite Value is situated. If no Opposite Type is specified, the Opposite Value resides on the x-axis. For example, $\hat{2}$ is on the x-axis. Different letters denote other axes, such as $2\hat{i}$ on the y-axis and $2\hat{j}$ on the z-axis.

5.3.4 Opposite Signs

The third component of an Opposite Value is the Opposite Sign, which indicates whether the Opposite Value is a $\hat{}$ or ^{v}. Opposite Values with the same Counter but different Opposite Signs cancel each other out when added. The symbol ? represents any Opposite Type, and the superscript $^?$ denotes any Opposite Sign.

An alphabetic character or term can represent an unknown Opposite Value or Counter.

5.3.5 Unitaries

Each Opposite Type has two unitaries, corresponding to each Opposite Sign. A unitary Opposite Value has a Counter of 1, and thus a magnitude of 1. The unitaries in R1 are $\hat{1}$ and 1^v. In R2, they are $\hat{1}, 1^v, \hat{i}$ and i^v. In R3, they are $\hat{1}, 1^v, \hat{i}, i^v, \hat{j}$ and j^v.

5.4 Dimensions

The Wave Number system encompasses an unlimited number of dimensional subsets, each representing a different number of dimensions, starting with one dimension. Each dimensional subset is denoted as Rn.

R1 is a one-dimensional subset based on the x-axis number line, with Opposite Values in the format $\hat{1}$, $\hat{2}$, etc.

R2 is a two-dimensional subset that builds on R1 by adding the y-axis number line, with Opposite Values in the format $\hat{\imath}$, $2\hat{\imath}$, etc.

R3 is a three-dimensional subset that builds on R2 by adding the z-axis number line, with Opposite Values in the format $\hat{\jmath}$, $2\hat{\jmath}$, etc.

R4 is a four-dimensional subset that builds on R3 by adding the w-axis number line, with Opposite Values in the format $\hat{k}$, $2\hat{k}$, etc.

R5 is a five-dimensional subset that builds on R4 by adding the v-axis number line, with Opposite Values in the format $\hat{l}$, $2\hat{l}$, etc.

Rn is an n-dimensional subset that builds on R$n-1$ by adding an additional axis number line, with Opposite Values in the format $\hat{?}$, $2\hat{?}$, etc.

5.5 Axes

All axes in an n-dimensional subset are orthogonal.

5.5.1 Opposite Values on Axes

An Opposite Value represents a point on a Wave Number axis or any finite or infinite decimal representation for the Opposite Type and Sign of the axis. Unlike classical mathematics, which uses real and imaginary numbers, the Wave Number system exclusively uses Opposite Values.

5.5.2 R1

The standard configuration of the x-axis is a line running from left to right with the origin at the center. An Opposite Value to the left of the origin is a ^{v} Opposite Value, while to the right of the origin, it is a $\hat{}$ Opposite Value.

Opposite Values in R1 include integers, fractions, and irrational Opposite Values, denoted as $\hat{1}, 1^v, \sqrt{2}\hat{}, \sqrt{2}^v, \hat{\pi}, \pi^v, \hat{e}$ and e^v etc.

5.5.3 R2

R2 consists of the x and y-axes. The standard configuration of the x-axis is a line running from left to right through the origin, while the y-axis runs from north to south through the origin. Opposite Values on the x-axis are the same as in R1. An Opposite Value south of the origin on the y-axis is an i^v Opposite Value, and one north of the origin is an $\hat{\imath}$ Opposite Value. Opposite Values in R2 include all R1 Opposite Values on the x-axis, as well as integers, fractions, and irrational Opposite Values on the y-axis, denoted as $\hat{\imath}, i^v, \sqrt{2}\hat{\imath}, \sqrt{2}i^v, \pi\hat{\imath}, \pi i^v, e\hat{\imath}$ and ei^v etc.

5.5.4 R3

R3 consists of the x, y, and z-axes. The standard configuration places the y-axis running from east to west (left to right) through the origin, the z-axis running from north to south through the origin, and the x-axis running from front to back through the origin. Opposite Values in R3 include all R2 Opposite Values on the x and y axes, as well as $\hat{\jmath}$ and j^v integers, fractions, and irrational Opposite Values on the z-axis, denoted as: $\hat{\jmath}, j^v, \sqrt{2}\hat{\jmath}, \sqrt{2}j^v, \pi\hat{\jmath}, \pi j^v, e\hat{\jmath}$ and ej^v etc.

An Opposite Value on the x-axis behind the origin is a ^{v} Opposite Value, while in front of the origin, it is a $\hat{\ }$ Opposite Value. On the y-axis, left of the origin, an Opposite Value is an i^v Opposite Value, while right of the origin, it is an $\hat{\imath}$ Opposite Value. On the z-axis, south of the origin, an Opposite Value is a j^v Opposite Value, while north of the origin, it is a $\hat{\jmath}$ Opposite Value. The standard orientation in R3 places $\hat{\jmath}$ on top and $\hat{\ }$ facing the viewer.

5.5.5 Higher Dimensions

In higher dimensions ($\mathrm{R}n, n > 3$), there are no standard configurations for the axes. Opposite Values in $\mathrm{R}n$ include all Opposite Values in $\mathrm{R}n - 1$ and a new range of Opposite Values based on the new axis. For example, R4 includes all R3 Opposite Values on the x, y, and z-axes, and adds $\hat{\mathrm{k}}$ and k^v integers, fractions, and irrational Opposite Values on the w-axis, denoted as: $\hat{\mathrm{k}}, k^v, \sqrt{2}\hat{k}, \sqrt{2}k^v, \pi\hat{k}, \pi k^v, e\hat{k}$ and ek^v etc.

5.6 Operations

5.6.1 Addition

The expression $\alpha + \beta$ represents the outcome of adding two Opposite Values α and β. This result is referred to as the sum of α and β.

5.6.2 Rotation $\circlearrowleft^?$

The second operation is rotation. $\alpha \circlearrowleft^? \beta$ denotes the result of rotating an Opposite Value β by the angle α. This result is referred to as the rotation of β by α. The angle α can be expressed in degrees or radians. When α has a $\hat{\ }$ Opposite Sign, $\alpha \circlearrowleft^? \beta$ represents a counterclockwise rotation, and when α has a ^{v} Opposite Sign, the operation performs a clockwise rotation.

The symbol $^?$ represents the point or axis of rotation indicated by Opposite Values. If the symbol is blank, assume the rotation is around an imaginary z-axis, a point at the origin (0) in R1 and (0, 0) in R2. If blank in R3, assume the rotation is around the z-axis (0, 0, $\hat{\jmath}$). When included, it represents a rotation around the point indicated in R1 and R2 and the axis indicated in R3.

5.6.3 Rotication $R \circlearrowleft^?$

The third operation is rotication. $\alpha\ R \circlearrowleft^? \beta$ denotes the rotication of an Opposite Value β by the angle α. This result is referred to as the rotication of β by α. Rotication consists of two steps. The first step is a rotation of β by α. The second step multiplies the result of this rotation by the magnitude of the axis of rotation.

5.6.4 Multiplication

The fourth operation is multiplication. $\alpha * \beta$ or $\alpha\beta$ denotes the result of multiplying two Opposite Values α and β. This result is referred to as the product of α and β. Multiplication is assumed when an Opposite Value precedes or follows a round bracket, or when two round brackets are adjacent.

5.6.5 Flipping

The final operation is flipping, which specifically rotates an Opposite Value α to its equivalent with the Opposite Sign changed. This corresponds to flipping from one side of the axis to the other. The superscript symbol $^{-}$ denotes the flipping operation. Consequently, flipping α results in the additive inverse of α, such that $\alpha + {}^{-}\alpha = 0$

The number 0 is special and does not have an Opposite Type or Sign. Flipping 0 makes no sense. However, calculations may yield ${}^{-}0$, in which case ${}^{-}0 = 0$.

Flipping necessitates the use of an alternative orthogonal axis to the one on which the point is located. This can be a real axis, such as the x-axis in R2 for a point on the y-axis, or the imaginary z-axis in R1 or R2.

5.7 General Definitions

1. The Opposite Values $2?^{?}$ to $10?^{?}$ are defined by $2?^{?} = 1?^{?} + 1?^{?}, 3?^{?} = 2?^{?} + 1?^{?}$, etc.
2. The Counters of other Opposite Values are defined by the normal rules of decimal representation.
3. The additive inverse of the Opposite Value α is ${}^{-}\alpha$ and means $\alpha +^{-} \alpha = 0$. Axiom 11 ensures the uniqueness and existence of the additive inverse.
4. $\alpha +^{-} \beta$ represents the Opposite Value that is the difference between α and β. It is calculated by adding the flip of β to α.
5. if $\alpha \neq 0$, the reciprocal of α in R1 and R2 is the Opposite Value α^{-1} that satisfies $\alpha * \alpha^{-1} = \hat{1}$, where $\alpha^{-1} = \hat{1}/\alpha$. Axiom 12 ensures the uniqueness and existence of the reciprocal in R1 and R2. The reciprocal is a multiplicative inverse in that multiplying by the reciprocal of an Opposite Value is the equivalent of dividing by the Opposite Value. Reciprocals of expressions exist, such as $(\hat{4} + 3i^{v})^{-1} = (0.16\hat{} + 0.12\hat{\imath})$.
6. If $\alpha \neq 0$, the reciprocal of α in R3 is the Opposite Value α^{-1} that satisfies $\alpha * \alpha^{-1} = 1 \ddagger \alpha^{\ddagger\alpha}$ where $\alpha^{-1} = (1 \ddagger \alpha^{\ddagger\alpha}/\alpha)$. Axiom 13 ensures the uniqueness and existence of the reciprocal in R3. The reciprocal of an Opposite Value is not a multiplicative inverse, which do not exist in R3. Reciprocals of expressions do not exist in R3.
7. if $\beta \neq 0$, the quotient of α and β, represented by α/β, is the Opposite Value obtained by dividing α by β.
8. An Opposite Value is considered to be rational with a rational Counter if it equates to p/q for a pair of integers p and q with $q \neq 0$.
9. If an Opposite Value is not rational, it is irrational and its Counter is irrational.
10. The expression α is less than or equal to β, represented by $\alpha <= \beta$, indicates that $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$.
11. The expression α is greater than β, represented by $\alpha > \beta$, indicates that $|\beta| < |\alpha|$.
12. The expression α is greater than or equal to β, represented by $\alpha >= \beta$, indicates that $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$.
13. The set of all $\hat{}$ Opposite Values in a dimension is represented by $R?\hat{}$, and the set of all $\hat{}$ integers by $Z?\hat{}$.
14. The set of all ^{v} Opposite Values in a dimension is represented by $R?^{v}$, and the set of all integers by $Z?^{v}$.
15. All Opposite Values, with the exception of 0, are greater than 0.

16. An Opposite Value α is said to be nonvee if $\alpha = 0$ or has a $\hat{}$ sign
17. An Opposite Value α is said to be nonhat if $\alpha = 0$ or has a ^{v} sign
18. If α, β and γ are distinct Opposite Values with equal Opposite Sign and Opposite Type, an Opposite Value γ is between α and β if $|\alpha| < |\gamma| < |\beta|$ or $|\alpha| > |\gamma| > |\beta|$.
19. The square of an Opposite Value α is the Opposite Value $\alpha * \alpha$, represented by α^2.
20. If Φ is a set of Opposite Values, an Opposite Value β is the largest element of Φ for that Opposite Type and Sign if β is an element of Φ with the same Opposite Type and Sign and $\beta >= x$ whenever x is any element of Φ for that Opposite Type and Sign. Therefore, there is a largest element of Φ for each Opposite Type and Sign. The definition of the term smallest element is similar.
21. If Φ is a set of Opposite Values, an Opposite Value β is said to be the largest absolute element of Φ if β is an element of Φ and $|\beta| >= |x|$ whenever x is any element of Φ. The definition of the term smallest absolute element is similar.
22. If Φ is a set of Opposite Values, an Opposite Value β (not necessarily in Φ) is said to be an upper bound for that Opposite Type and Sign in Φ if β has the same Opposite Type and Sign and $\beta >= x$ for every x with that Opposite Type and Sign in Φ. It is said to be a least upper bound for that Opposite Type and Sign in Φ if every other upper bound β' for Φ satisfies $\beta' >= \beta$. The definitions of the terms lower bound and greatest lower bound are similar.
23. If Φ is a set of Opposite Values, an Opposite Value β (not necessarily in Φ) is said to be an absolute upper bound for Φ if $|\beta| >= |x|$ for every x in Φ. It is said to be a least absolute upper bound Φ if every other upper bound β' for Φ satisfies $|\beta'| >= |\beta|$. The terms absolute lower bound and absolute lowest bound are defined similarly.

5.8 Equality

5.8.1 Introduction

The "Axioms for Real Numbers" states the following about equality[7]:

"In modern mathematics, the relation 'equals' can be used between any two 'mathematical objects' of the same type, such as numbers, matrices, sets, functions, etc. To say that $\alpha = \beta$ is simply to say that the symbols α and β represent the very same object. Thus, equality really belongs to logic rather than to any particular branch of mathematics. Equality always has the following fundamental properties, no matter what kinds of objects it is applied to."

5.8.2 General Properties of Equality

1. Reflexivity: $\alpha = \alpha$.
2. Symmetry: If $\alpha = \beta$, then $\beta = \alpha$.
3. Transitivity: If $\alpha = \beta$ and $\beta = \gamma$, then $\alpha = \gamma$.
4. Substitution: If $\alpha = \beta$, then any or all occurences of α in a mathematical expression can be replaced with β without changing the value of the expression.

5.8.3 Properties of Equality of Opposite Values

The following properties of Equality also exist for Opposite Values. The first five statements assert, in essence, that starting from a true equation between two Opposite Values, one can perform the same operation on both sides and still maintain the validity of the equation. The final two statements say that, given two true equations, combining them, whether by addition, multiplication, or division (as long as division by zero is avoided), will result in another true equation. Each of these propositions can be demonstrated using only the reflexive property of equality and the principle of substitution. In these statements, α, β, γ and δ denote arbitrary opposing values.

1. If $\alpha = \beta$, then $\alpha + \gamma = \beta + \gamma, \alpha\gamma = \beta\gamma$, and $\alpha +^- \gamma = \beta +^- \gamma$.
2. If $\alpha = \beta$ and γ is nonzero, then $\alpha/\gamma = \beta/\gamma$.
3. If $\alpha = \beta$, then $^-\alpha =^- \beta$.
4. If $\alpha = \beta$ and α and β are both nonzero, then $\alpha + 1?^? = \beta + 1?^?$.
5. If $\alpha = \beta$, then $\alpha^2 = \beta^2$.
6. If $\alpha = \beta$ and $\gamma = \delta$, then $\alpha + \gamma = \beta + \delta, \alpha\gamma = \beta\delta$, and $\alpha +^- \gamma = \beta +^- \delta$.
7. If $\alpha = \beta$ and $\gamma = \delta$, and γ and δ are both nonzero, then $\alpha/\gamma = \beta/\delta$.

5.9 Rotation

5.9.1 Introduction

Rotation has no equivalent in the "Axioms for Real Numbers."[7] As a fundamental operation, rotation necessitates its own definition. This leads to the following rules of rotation:

A rotation moves a point, represented by Opposite Value(s), to a new location, also represented by Opposite Value(s), through a circular movement around an axis, which is similarly represented by Opposite Value(s). At least one point remains fixed during the rotation.

5.9.2 Return Rule

The first rule of rotation, known as the Return Rule, states that continuous rotation of a point by the same angle will return the point to its original location when the total rotation equals a multiple of 360^o.These rotational paths are referred to as Return Rings. A point can move to any other axis, provided it is not rotating around its own axis.

5.9.3 Orthogonal Rule

The second rule of rotation, the Orthogonal Rule, states that multiplication by a unitary is equivalent to a rotation around the unitary's axis by 90^o. Rotation by an Opposite Value with a $\hat{}$ sign is in a counterclockwise direction, while rotation by an Opposite Value with a ^{v} sign is in a clockwise direction. The sign used is the sign of the unitary.

Unitary rotation in R1 is non-standard since there is no orthogonal axis to the x-axis for rotation. Rotation is considered around a single, dimensionless point.

Consider multiplication by the unitary $1^{\wedge}$ as a rotation around a dimensionless point on the z-axis by $90^{\wedge o}$. Consequently, the point located at $1^{\wedge}$ would rotate counter-clockwise to $i^{\wedge}$ on an imaginary y-axis. However, since $i^{\wedge}$ does not exist in R1, $1^{\wedge}$ cannot undergo a $90^{\wedge o}$ rotation to $i^{\wedge}$, and thus, multiplication by the unitary $1^{\wedge}$ does not result in any point movement.

In contrast, multiplication by the unitary 1^v is conceptualized as a rotation around a dimensionless point on the z-axis by 90^{vo}. As a result, the point at $1^{\wedge}$ would rotate clockwise to i^v on an imaginary y-axis. Though i^v does not exist in R1 and $1^{\wedge}$ cannot be rotated by 90^{vo} to it , multiplication by the unitary 1^v does cause point movement. It effectively moves a point from 1^v to $1^{\wedge}$ and from $1^{\wedge}$ to 1^v, corresponding to a 180^o rotation, which is equivalent to flipping. The presence of only one axis distorts rotations in R1.

Unitary rotation in R2 is non-standard since there is no axis orthogonal to the x-y plane. Rotation is considered around a single point.

Unitary rotation by $i^{\wedge}$ and i^v adheres to the Orthogonal Rule. Given that $i^{\wedge}$ and i^v follow the Orthogonal Rule, it can be deduced that no rotation occurs when multiplying by the unitary $1^{\wedge}$, and a 180^o rotation occurs when multiplying by the unitary 1^v.

Unitary rotation is standard in R3, therefore the Orthogonal Rule applies in full.

5.9.4 Reversal Rule

The third rule of rotation, the Reversal Rule, states that a rotation reverses by rotating the result of the first rotation by the Operator of the first rotation with the sign reversed.

In other words, a rotation in one direction reverses by doing the same rotation in the opposite direction. In cases where a point rotates around its own axis, no actual rotation occurs, but the Reversal Rule still applies.

The Reversal Rule does not apply in R1 and R2 when multiplying by $1^{\wedge}$ due to the absence of an orthogonal axis of rotation. This results in no rotation when multiplying by the unitary $1^{\wedge}$ and a 180^o rotation when multiplying by the unitary 1^v. Therefore, $1^{\wedge} * 1^{\wedge} = 1^{\wedge}$ and $1^v * 1^{\wedge} = 1^v$.

The Reversal Rule fully applies in R3.

5.9.5 Remain Rule

The fourth rule of rotation, known as the Remain Rule, states that a point remains in the same location when rotated about its own axis.

5.9.6 Zero Rules

The fifth rule of rotation, known as the Zero Rule for R3, states:

1. $\alpha * \beta + \alpha *^{-} \beta = 0$.

2. $\alpha * \beta + \beta * \alpha = 0$ provided α and β are not of the same Opposite Type and Opposite Sign.
3. $\alpha + \beta(\beta * \alpha) = 0$ provided α and β are unitaries and not of the same Opposite Type and Opposite Sign.

5.10 General Axioms

The following axioms are adapted from the 15 axiom statements in "Axioms for Real Numbers"[7], specifically interpreted for Wave Numbers.

Wave Numbers presuppose the truth of the following statements, where α, β, γ and δ denote arbitrary Opposite Values.

1. Existence: A set R? consisting of all Opposite Values exists for the dimension ?. It contains a subset $Z? \subseteq R?$ consisting of all integers for the dimension ?.
2. Closure of Z?: If α and β are integers, then both $\alpha + \beta$ and $\alpha * \beta$ are also integers.
3. Closure of R?: If α and β are Opposite Values, then both $\alpha + \beta$ and $\alpha * \beta$ are also Opposite Values.
4. Commutativity in R1 and R2:
   - $\alpha + \beta = \beta + \alpha$
   - $\alpha\beta = \beta\alpha$
   - $|\alpha\beta| = |\beta\alpha|$
5. Commutativity in R3 and higher dimensions:
   - $\alpha + \beta = \beta + \alpha$
   - $\alpha\beta \neq \beta\alpha$
   - $|\alpha\beta| = |\beta\alpha|$
6. Associativity in R1 and R2: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all Opposite Values α, β, and γ in R1 and R2.
7. Associativity in R3 and higher dimensions:
   - $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all Opposite Values α, β, and γ in R3 and higher dimensions.
   - $(\alpha\beta)\gamma \neq \alpha(\beta\gamma)$ for all Opposite Values α, β, and γ in R3 and higher dimensions.
8. Distributivity: $\alpha(\beta + \gamma) = \alpha * \beta + \alpha * \gamma$ for all consolidated Opposite Values α, β, and γ. Note a consolidated Opposite Value means that any Opposite Values of the same Opposite Type have been added together
9. Zero: The Opposite Value 0 is an integer that conforms to the equation $(\alpha + 0) = \alpha = 0 + \alpha$ for every Opposite Value α.
10. One: The Counter 1 is a Counter that that conforms to the equation $(1 * \alpha) = \alpha$ for every Opposite Value α.
11. Additive inverses: If α is any Opposite Value, then there is a unique Opposite Value $^{-}\alpha$ such that $\alpha +^{-} \alpha = 0$. This is the Opposite Value with the same Counter and just the Opposite Sign changed.
12. Reciprocals in R1 and R2: If α is any nonzero Opposite Value, then there is a unique Opposite Value α^{-1} such that $\alpha * \alpha^{-1} = \hat{1}$.

13. Reciprocals in R3: If α is any nonzero Opposite Value, then there is a unique Opposite Value α^{-1} such that $\alpha * \alpha^{-1} = 1 \ddagger \alpha^{\ddagger\alpha}$.
14. Trichotomy law: If α and β are Opposite Values, then one and only one of the following three statements is true: $|\alpha| < |\beta|, |\alpha| = |\beta|$, or $|\alpha| > |\beta|$.
15. Closure of R:
    - In R1 and R2, if α and β are Opposite Values of the $\hat{}/^{v}$ Opposite Type and $\hat{}$ Opposite Sign , then $\alpha + \beta$ and $\alpha * \beta$ are of the $\hat{}/^{v}$ Opposite Type and $\hat{}$ Opposite Sign.
    - In R3, if α and β are Opposite Values of the same Opposite Type, then $\alpha * \beta$ is of the same Opposite Type and Sign as the Operand.
    - In R3, if α and β are Opposite Values of the same Opposite Type and Sign, then $\alpha + \beta$ is of the same Opposite Type and Sign.
16. Addition law for inequalities: If α, β, and γ are Opposite Values and $|\alpha| < |\beta|$, then $|\alpha| + |\gamma| < |\beta| + |\gamma|$.
17. The well ordering axiom: There is a smallest element in every nonempty set of $?^{?}$ integers in R?.
18. The least upper bound axiom: If Φ is any nonempty set of Opposite Values in R? and Φ has an upper bound for an Opposite Type and Sign, then Φ has a least upper bound for that Opposite Type and Sign.

5.11 Theorems

5.11.1 Introduction

The theorems in this section are derived from the theorems selected in "Axioms for Real Numbers."[7] as interpreted for Wave Numbers.

These theorems can be proved from the axioms in the order listed. In all statements, α, β, γ and δ represent arbitrary Opposite Values.

5.11.2 Zero

1. $\alpha +^{-} \alpha = 0$.
2. $0 +^{-} \alpha =^{-} \alpha$.
3. $0 * \alpha = 0$ and $\alpha * 0 = 0$.
4. If $\alpha * \beta = 0$, then $\alpha = 0$ and/or $\beta = 0$.

5.11.3 Flipping

1. $^{-}0 = 0$.
2. $^{-}(^{-}\alpha) = \alpha$.
3. R1 and R2: $^{-}\alpha * \beta =^{-} (\alpha * \beta)$
4. R3 and higher: $^{-}\alpha * \beta \neq^{-} (\alpha * \beta)$
5. R1 and R2: $^{-}\alpha *^{-} \beta = \alpha * \beta$
6. R3 and higher: $^{-}\alpha *^{-} \beta \neq \alpha * \beta$
7. $^{-}\alpha?^{?} = |\alpha| *^{-} 1 \ddagger \alpha^{\ddagger\alpha}$
8. $^{-}\alpha?^{\hat{}} = \alpha?^{v};^{-} \alpha?^{v} = \alpha?^{\hat{}}$

5.11.4 Distributive

1. $^-(\alpha+\beta) = (^-\alpha) + (^-\beta) =^- \alpha +^- \beta$.
2. $^-(\alpha +^- \beta) =^- \alpha + \beta$.
3. $^-(^-\alpha +^- \beta) = \alpha + \beta$.
4. $\alpha + \alpha = 2\alpha$.
5. In R1 and R2: $\alpha(\beta +^- \gamma) = \alpha * \beta +^- \alpha * \gamma = (\beta +^- \gamma)\alpha$. Note that $\alpha(\beta +^- \gamma)$ is the equivalent of $\alpha * (\beta +^- \gamma)$ and $(\beta +^- \gamma)\alpha$ the equivalent of $(\beta +^- \gamma) * \alpha$ as multiplication with round brackets is defined earlier.
6. In R3: $\alpha(\beta +^- \gamma) = \alpha * \beta + \alpha *^- \gamma \neq (\beta +^- \gamma)\alpha$.
7. $(\alpha+\beta)(\gamma+\delta) = \alpha * \gamma + \alpha * \delta + \beta * \gamma + \beta * \delta$.
8. In R1 and R2: $(\alpha+\beta)(\gamma + -\delta) = \alpha\gamma + \alpha^-\delta + \beta\gamma + \beta^-\delta = (\gamma +^- \delta)(\alpha+\beta)$.
9. In R3: $(\alpha+\beta)(\gamma +^- \delta) = \alpha\gamma + \alpha^-\delta + \beta\gamma + \beta^-\delta \neq (\gamma +^- \delta)(\alpha+\beta)$.
10. $(\alpha +^- \beta)(\gamma +^- \delta) = \alpha\gamma + \alpha^-\delta +^- \beta\gamma + \beta\delta$.

5.11.5 Reciprocals and Multiplicative Inverses

1. Multiplicative inverses exist in R1 and R2 but not in R3.
2. If α is nonzero, then so is α^{-1}.
3. In R1 and R2: $\alpha^{-1} = \hat{1}/ \ddagger \alpha^{\ddagger\alpha}$.
4. In R3: $\alpha^{-1} = 1 \ddagger \alpha^{\ddagger\alpha}/\alpha$.
5. $(\alpha^{-1})^{-1} = \alpha$ if α is nonzero.
6. $|1?^{?-1}| = 1$ and is a Counter that cannot be used standalone.
7. $(^-\alpha)^{-1} =^- (\alpha^{-1})$ if α is nonzero.
8. $(\alpha\beta)^{-1} = \alpha^{-1}\beta^{-1}$ if α and β are nonzero.
9. In R1 and R2: $(\alpha/\beta)^{-1} = \beta/\alpha$ if α and β are nonzero. Note that in R3 it is not true.

5.11.6 Quotients

1. Division by 0 is undefined
2. Division is not associative.
3. $\alpha/1 = \alpha$.
4. R1 and R2: $\hat{1}/\alpha = \alpha^{-1}$ if α is nonzero.
5. R3: $1 \ddagger \alpha^{\ddagger\alpha}/\alpha = \alpha^{-1}$ if α is nonzero.
6. R1 and R2: $\alpha/\alpha = \hat{1}$ if α is nonzero.
7. R3: $\alpha/|\alpha\lceil = \alpha/|\alpha|^{v} = 1 \ddagger \alpha^{\ddagger\alpha}$ if α is nonzero.
8. In R2: $(\alpha/\beta)(\gamma/\delta) = (\alpha * \gamma)/(\beta * \delta)$ if β and δ are nonzero. Note that in R3 it is not true as multiplication and division is not associative
9. In R1 and R2:$(\alpha/\beta)/(\gamma/\delta) = (\alpha * \delta)/(\beta * \gamma)$ if β, γ and δ are non-zero. Note that in R3 it is not true as multiplication and division are not associative.
10. In R1 and R2: $(\alpha/\beta)/(\gamma/\delta) = (\delta * \alpha)/(\beta * \gamma)$ if β, γ and δ are nonzero. Note that in R3 it is not true as multiplication and division are not associative.
11. In R1 and R2: $(\alpha * \gamma)/(\beta * \gamma) = \alpha/\beta$ if β and γ are nonzero. Note that in R3 it is not true as multiplication and division are not associative
12. In R1 and R2: $\alpha(\beta/\gamma) = (\alpha * \beta)/\gamma$ if γ is nonzero. Note that in R3 it is not true as multiplication and division are not associative.

13. In R1 and R2:$(\alpha * \beta)/\beta = \alpha$ if β is nonzero. Note that in R3 it is not true as multiplication and division are not associative.
14. $(^-\alpha)/\beta =^- (\alpha/\beta)$ if β is nonzero.
15. $(^-\alpha)/(^-\beta) = \alpha/\beta$ if β is nonzero.
16. In R1 and R2: $\alpha/\beta + \gamma/\delta = (\alpha * \delta + \beta * \gamma)/(\beta * \delta)$ if β and δ are nonzero. Note that in R3 it is not true as multiplication and division are not associative.
17. In R1 and R2: $\alpha/\beta + \gamma/\delta = (\delta * \alpha + \gamma * \beta)/(\beta * \delta)$ if β and δ are nonzero. Note that in R3 it is not true as multiplication and division are not associative.
18. In R1 and R2: $\alpha/\beta +^- \gamma/\delta = (\alpha * \delta + \beta *^- \gamma)/(\beta * \delta)$ if β and δ are nonzero. Note that in R3 it is not true as multiplication is not associative.
19. In R1 and R2: $\alpha/\beta +^- \gamma/\delta = (d * \alpha +^- \gamma * \beta)/(\beta * \delta)$ is true if β and δ are nonzero. Note that in R3 it is not true as multiplication is not associative.
20. In R2: $(\alpha + \beta i)/(\alpha + \beta i) = \hat{1}$
21. If $\alpha/\beta = \gamma$, then $\beta * \gamma = \alpha$

5.11.7 Squares

1. For every $\alpha, \alpha^2 >= 0$.
2. $\alpha^2 = 0$ if and only if $\alpha = 0$.
3. $\alpha^2 > 0$ if $\alpha \neq 0$.
4. $(^-\alpha)^2 =^- \alpha *^- \alpha = \alpha * \alpha = \alpha^2$.
5. R1 and R2: $(\alpha^{-1})^2 = \hat{1}/\alpha^2$.
6. R3: $(\alpha^{-1})^2 = 1 \ddagger \alpha^{\ddagger\alpha}/\alpha^2$.
7. R1 and R2: if $\alpha^2 = \beta^2$, then $\alpha = \beta$ or $\alpha =^- \beta$.
8. R3: if $\alpha^2 = \beta^2$, then $\alpha = \beta$.
9. If $\alpha < \beta \implies \alpha^2 < \beta^2$.

5.11.8 Inequalities

Transitivity

1. if $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.
2. If $\alpha <= \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.
3. If $\alpha < \beta$ and $\beta <= \gamma$, then $\alpha < \gamma$.
4. If $\alpha <= \beta$ and $\beta <= \gamma$, then $\alpha <= \gamma$.

Other Properties

1. If α is non-zero, $0 < \alpha$.
2. If $\alpha <= \beta$ and $\beta <= \alpha$, then $|\alpha| = |\beta|$.
3. If $\alpha < \beta$, then $^-\alpha <^- \beta$.
4. $0 <\hat{?}$ and $0 <?^v$.
5. If α is non-zero and $\alpha < \beta$, then $\alpha^{-1} > \beta^{-1}$.
6. If $\alpha < \beta$ and $\gamma < \delta$, then $\alpha + \gamma < \beta + \delta$.
7. If $\alpha <= \beta$ and $\gamma < \delta$, then $\alpha + \gamma < \beta + \delta$.
8. If $\alpha <= \beta$ and $\gamma <= \delta$, then $\alpha + \gamma <= \beta + \delta$.
9. If γ is non-zero and $\alpha < \beta$, then $\alpha\gamma < b\gamma$.
10. $\alpha < \beta$ and $\gamma < 0$ are not valid equations as no Opposite Value < 0.

11. If $\alpha <= \beta$, then $\alpha\gamma <= b\gamma$.
12. If $\alpha < \beta$ and $\gamma < d$, then $\alpha\gamma < \beta\delta$.
13. If $\alpha <= \beta$ and $\gamma <= d$, then $\alpha\gamma <= \beta\delta$.
14. $\alpha\beta > 0$ if neither α nor $\beta = 0$.
15. $\alpha\beta < 0$ is not a valid equation.
16. There does not exist a smallest $\hat{}$ Opposite Value.
17. There does not exist a smallest ^{v} Opposite Value.
18. The smallest Opposite Value is 0.
19. Density: There exist infinitely many rational Counters and infinitely many irrational Counters between α and β, if they are distinct Counters.

5.11.9 Order Property of Integers

1. $1\hat{?}$ is the smallest $\hat{?}$ integer.
2. $1?^v$ is the smallest $?^v$ integer.
3. If $m?^?$ and $n?^?$ are integers such that $m?^? > n?^?$, then $m?^? >= n?^? + 1?^?$.
4. There does not exist a largest integer.

6 Wave Number Operations

6.1 Addition

In R3, addition in the Wave Number system is straightforward. Opposite Values with Opposite Signs cancel each other out through interference. See the examples below:

1. $\hat{1} + 0 = \hat{1}; \hat{\imath} + 0 = \hat{\imath}; \hat{\jmath} + 0 = \hat{\jmath}$.
2. $\hat{1} + \hat{\imath} + \hat{\jmath} = \hat{1} + \hat{\imath} + \hat{\jmath}$.
3. $1^v + 0 = 1^v; i^v + 0 = i^v; j^v + 0 = j^v$.
4. $\hat{1} + 1^v = 0; \hat{\imath} + i^v = 0; \hat{\jmath} + j^v = 0$.
5. $1^v + \hat{1} = 0; i^v + \hat{\imath} = 0; j^v + \hat{\jmath} = 0$.
   Note that addition is commutative.
6. $1^v + i^v + j^v = 1^v + i^v + j^v$
7. $0 + 0 = 0$.
   Note 0 is neither $\hat{}$ or ^{v}.
8. $\hat{4} + \hat{6} = \hat{10}; 4\hat{\imath} + 6\hat{\imath} = 10\hat{\imath}; 4\hat{\jmath} + 6\hat{\jmath} = 10\hat{\jmath}$.
9. $\hat{4} + \hat{6} + 3^v = \hat{7}; 4\hat{\imath} + 6\hat{\imath} + 3i^v = 7\hat{\imath}; 4\hat{\jmath} + 6\hat{\jmath} + 3j^v = 7\hat{\jmath}$.
10. $4^v + \hat{6} + 3^v = 1^v; 4i^v + 6\hat{\imath} + 3i^v = i^v; 4j^v + 6\hat{\jmath} + 3j^v = j^v$.
11. $(4^v + \hat{6}) + 3^v = 1^v; (4i^v + 6\hat{\imath}) + 3i^v = i^v; (4j^v + 6\hat{\jmath}) + 3j^v = j^v$.
12. $4^v + (\hat{6} + 3^v) = 1^v; 4i^v + (6\hat{\imath} + 3i^v) = i^v; 4j^v + (6\hat{\jmath} + 3j^v) = j^v$.
    Note that addition is associative.
13. $4^v + 6\hat{\imath} + 3j^v = 4^v + 3j^v + 6\hat{\imath} = 3j^v + 4^v + 6\hat{\imath}$.
    Order makes no difference.

6.2 Rotation

6.2.1 Introduction

What is Rotation in R3?

The R3 rotation operation is a fundamental aspect of the Wave Number system. The facts expressed in its axioms support this. A rotation moves a point, represented by Opposite Values, to a new location, also represented by Opposite Values, through circular movement around an axis. This axis is represented by a pair of Opposite Values. It remains fixed during the rotation.

Typically, one of the pair of Opposite Values that defines the axis is the origin $(0, 0, 0)$, but any pair of Opposite Values can define an axis.

Syntax

Rotation in R3 requires an amount of rotation in degrees or radians, an axis, and a starting point. It is expressed as follows, where $^?$ represents the axis of rotation. The default axis is the z-axis, described by the pair of Opposite Values $(0, 0, 0)$ and $(0, 0, \hat{\jmath})$:

$$\text{(Amount of rotation)} \circlearrowleft^{?} \text{(Opposite Value)}$$

In sample calculations, formulae, and the calculator, the axis is assumed to be formed by the origin and the given Opposite Value. If an axis does not pass through the origin, it needs to be translated to the origin, the rotation performed, and then the result translated back by reversing the original translation.

Examples of R3 Rotation

Consider the expression $a \circlearrowleft b$. Here, b is the operand, a is the operator, and $\circlearrowleft$ denotes the rotation operation.

For $\frac{\pi}{2}^{\wedge} \circlearrowleft 1^{\wedge} = \hat{\imath}$:

- This means taking a point located at $1^{\wedge}$ on the x-axis and rotating it counterclockwise by $90^{\hat{o}}$ around the z-axis defined by $(0, 0, 0)$ and $(0, 0, \hat{\jmath})$.
- The point moves from $(1^{\wedge}, 0, 0)$ to $(0, \hat{\imath}, 0)$, placing it at $\hat{\imath}$ on the y-axis.

For $\frac{\pi}{2}^{v} \circlearrowleft 1^{\wedge} = i^{v}$:

- This means taking a point located at $1^{\wedge}$ on the x-axis and rotating it clockwise by $90^{v\,o}$ around the z-axis defined by $(0, 0, 0)$ and $(0, 0, j^{v})$.
- The point moves from $(1^{\wedge}, 0, 0)$ to $(0, i^{v}, 0)$, placing it at i^{v} on the y-axis.

Pythagoras' Theorem

To calculate the distance from the origin, or the radius, of any point (x, y, z), use Pythagoras' theorem:

Distance $= \sqrt{|x^2| + |y^2| + |z^2|}$

6.2.2 Rotation and Multiplication Table

The layout of the R3 axes is defined as part of the Wave Number axioms. The top view of a Wave Number sphere is illustrated in Figure 5. The Axioms of Rotation provide the Orthogonal Rule, which states that multiplication by a unitary is equivalent to

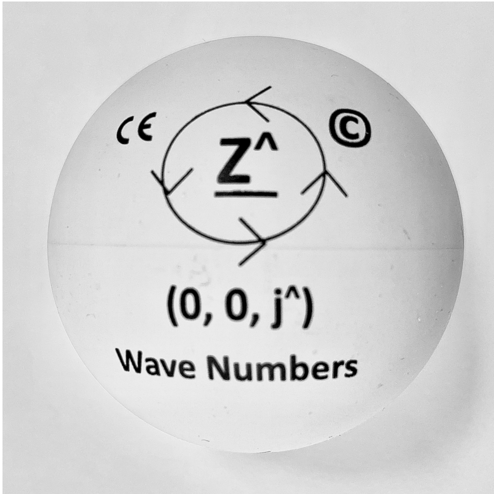

Fig. 5 Wave Number Sphere

Table 1 R3 rotation/multiplication table

$*$	$\hat{1}$	1^v	$\hat{\imath}$	i^v	$\hat{\jmath}$	j^v
$\hat{1}$	$\hat{1}$	1^v	$\hat{\jmath}$	j^v	i^v	$\hat{\imath}$
1^v	$\hat{1}$	1^v	j^v	$\hat{\jmath}$	$\hat{\imath}$	i^v
$\hat{\imath}$	j^v	$\hat{\jmath}$	$\hat{\imath}$	i^v	$\hat{1}$	1^v
i^v	$\hat{\jmath}$	j^v	$\hat{\imath}$	i^v	1^v	$\hat{1}$
$\hat{\jmath}$	$\hat{\imath}$	i^v	1^v	$\hat{1}$	$\hat{\jmath}$	j^v
j^v	i^v	$\hat{\imath}$	$\hat{1}$	1^v	$\hat{\jmath}$	j^v

rotation around the unitary's axis by 90^o. This allows for the derivation of the R3 unitary multiplication table, shown in Table 1, from rotations on the sphere.

6.2.3 Unitary Rotation/Multiplication

The Orthogonal Rule from the Axioms of Rotation states that multiplication by a unitary is equivalent to rotation around the unitary's axis by 90^o.

Some examples of unitary multiplications follow.

- $\frac{\hat{\pi}}{2} \circlearrowleft^z (\hat{1}, 0, 0) = \hat{\imath}$ or in multiplication $\hat{\jmath} * \hat{1} = \hat{\imath}$.
- $\frac{\hat{\pi}}{2} \circlearrowleft^y (\hat{1}, 0, 0) = j^v$ or in multiplication $\hat{\imath} * \hat{1} = j^v$.
- $\frac{\hat{\pi}}{2} \circlearrowleft^x (\hat{1}, 0, 0) = \hat{1}$ or in multiplication $\hat{1} * \hat{1} = \hat{1}$.
- $\frac{\pi^v}{2} \circlearrowleft^z (\hat{1}, 0, 0) = i^v$ or in multiplication $j^v * \hat{1} = i^v$.
- $\frac{\pi^v}{2} \circlearrowleft^z (\hat{2}, 3i^v, 4j^v) = (3^v, 2i^v, 4j^v)$ or in multiplication $j^v * (\hat{2} + 3i^v + 4j^v) = (3^v + 2i^v + 4j^v)$.

6.2.4 Wave Number Rotation Formula

Up to this point, we've focused on unitary rotations around a single axis on the unit sphere. Now, let's delve into more complex rotations in R3, where the operand can be any combination of Opposite Values with any degree of rotation. This section introduces the Wave Number Rotation formula, based on Rodrigues' formula for calculating rotation: [12].

$$v_{rot} = cos\theta(v) + sin\theta(u_r \times v) + (1 - cos\theta)(u_r \cdot v)u_r$$

In this formula:

- v is the vector being rotated.
- u_r is the unit vector representing the axis of rotation.
- θ is the angle of rotation.

This approach allows for the calculation of rotations around any vector through the origin, effectively handling complex rotations in $\mathbb{R}3$.

Rewrite the v_{rot} formula as follows:

- $v_{rot} = cos\theta(v) + sin\theta(u_r \times v) + (u_r \cdot v)u_r - cos\theta(u_r \cdot v)u_r$
- Rewrite $cos\theta(v)$ as $cos\theta(u_r \cdot u_r)v$ because $(u_r \cdot u_r) = u_r^2 = 1$ since u_r is the unit vector.
- $v_{rot} = cos\theta(u_r \cdot u_r)v + sin\theta(u_r \times v) + (u_r \cdot v)u_r - cos\theta(u_r \cdot v)u_r$
- The vector triple product rule [9] states that $(a \times b) \times c = (a \cdot c)b - (a \cdot b)c$.
- So, $cos\theta(u_r \cdot u_r)v - cos\theta(u_r \cdot v)u_r = cos\theta((u_r \times v) \times u_r)$
- giving
- $v_{rot} = (u_r \cdot v)u_r + sin\theta(u_r \times v) + cos\theta((u_r \times v) \times u_r)$

Wave Numbers uses this formula to calculate rotation in R3.

6.2.5 Example of Single Rotation

Take, for example, the rotation $60\hat{^o} \circlearrowleft^{(4\hat{\ }+5\hat{\imath}+3\hat{\jmath})} (1\hat{\ } + 2\hat{\imath} + 3\hat{\jmath})$.

$$v_{rot} = (\begin{bmatrix} 0.456\hat{\ } \\ 0.571\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} \cdot \begin{bmatrix} 1\hat{\ } \\ 2\hat{\imath} \\ 3\hat{\jmath} \end{bmatrix}) \begin{bmatrix} 0.456\hat{\ } \\ 0.571\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} + sin60\hat{^o}(\begin{bmatrix} 0.456\hat{\ } \\ 0.571\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 1\hat{\ } \\ 2\hat{\imath} \\ 3\hat{\jmath} \end{bmatrix}) + cos60\hat{^o}((\begin{bmatrix} 0.456\hat{\ } \\ 0.571\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 1\hat{\ } \\ 2\hat{\imath} \\ 3\hat{\jmath} \end{bmatrix}) \times \begin{bmatrix} 0.456\hat{\ } \\ 0.571\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix})$$

$$v_{rot} = 3.648 \begin{bmatrix} 0.456\hat{\ } \\ 0.571\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} + 0.866(\begin{bmatrix} 0.342\hat{\ } \\ 0.684i^v \\ 0.342\hat{\jmath} \end{bmatrix}) + 0.5(\begin{bmatrix} 0.342\hat{\ } \\ 0.684i^v \\ 0.342\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 0.456\hat{\ } \\ 0.571\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix})$$

$$
v_{rot} = \begin{bmatrix} 1.662\hat{\ } \\ 2.078\hat{\imath} \\ 2.494\hat{\jmath} \end{bmatrix} + \begin{bmatrix} 0.296\hat{\ } \\ 0.592i^{v} \\ 0.296\hat{\jmath} \end{bmatrix} + \begin{bmatrix} 0.331^{v} \\ 0.39i^{v} \\ 0.253\hat{\jmath} \end{bmatrix}
$$

$$
v_{rot} = \begin{bmatrix} 1.627\hat{\ } \\ 1.447\hat{\imath} \\ 3.043\hat{\jmath} \end{bmatrix}
$$

Equivalent $\mathbb{R}3$ Rotation

Using the Vector Rotation functionality of the Omni Quaternion Calculator[13] for an equivalent rotation in $\mathbb{R}3$, the point at (1, 2, 3) rotated by 60^{o} around (4, 5, 6) moves to (1.627, 1.447, 3.043). This matches the Wave Number rotation result obtained above.

Equivalent Quaternion Rotation

Pure quaternions are quaternions without a scalar component. The $\mathbb{R}3$ vector equivalent in classical mathematics for the axis $(4\hat{\ }+5\hat{\imath}+3\hat{\jmath})$ is (4, 5, 6), and for the Opposite Value rotated $(1\hat{\ }+2\hat{\imath}+3\hat{\jmath})$, it is (1, 2 ,3).

Quaternions represent the rotation as follows:

- $(4, 5, 6)$ is $(4i + 5j + 3k)$
- The unit vector of the axis of rotation u_r is $(4i + 5j + 3k)/\sqrt{(4^2 + 5^2 + 6^2)} = (0.456i + 0.57j + 0.683k)$
- The vector to be rotated, q_v at $(1, 2, 3)$ is $(1i + 2j + 3k)$
- The following formula gives the quaternion of rotation, q_r, for a given unit vector (x, y, z) and angle of rotation θ [9]:
- $q_r = cos(\frac{\theta}{2}) + (xi + yj + zk) * sin(\frac{\theta}{2})$
- So, $q_r = cos(\frac{60^o}{2}) + (0.456i + 0.57j + 0.683k) * sin(\frac{60^o}{2})$
- $= (0.866 + 0.228i + 0.285j + 0.342k)$
- $q_r * q_v$ rotates the vector by the quaternion of rotation.
- $(0.866+0.228i+0.285j+0.342k)(1i+2j+3k) = (-1.824+1.037i+1.39j+2.769k)$

To convert a quaternion back to an $\mathbb{R}3$ vector in classical maths, multiply the quaternion with the conjugate of the quaternion of rotation which is $(0.866 - 0.228i - 0.285j - 0.342k)$[9].

- $(-1.824 + 1.037i + 1.39j + 2.769k)(0.866 - 0.228i - 0.285j - 0.342k)$
- $= (1.627i + 1.447j + 3.043k)$

This matches the Wave Number rotation result obtained above.

6.2.6 Example of a Chain of Rotations

This section provides an example of a rotation chain involving three consecutive rotations.

- Rotation 1: $120^{\hat{o}} \circlearrowleft^{(\hat{1},\hat{2},\hat{1})} (1^v + i^v + \hat{\jmath})$
- Rotation 2: $45^{\hat{o}} \circlearrowleft^{(\hat{1},\hat{2},\hat{3})}$ (Result of Rotation 1)
- Rotation 3: $60^{\hat{o}} \circlearrowleft^{(\hat{4},\hat{5},\hat{6})}$ (Result of Rotation 2)

Calculate the chain of rotations using the Wave Number v_rot formula.

Rotation 1:

$$v_{rot} = (\begin{bmatrix} 0.40\hat{8} \\ 0.817\hat{\imath} \\ 0.408\hat{\jmath} \end{bmatrix} \cdot \begin{bmatrix} 1^v \\ i^v \\ \hat{\jmath} \end{bmatrix}) \begin{bmatrix} 0.40\hat{8} \\ 0.817\hat{\imath} \\ 0.408\hat{\jmath} \end{bmatrix} + sin120^{\hat{o}}(\begin{bmatrix} 0.40\hat{8} \\ 0.817\hat{\imath} \\ 0.408\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 1^v \\ i^v \\ \hat{\jmath} \end{bmatrix}) + cos120^{\hat{o}}((\begin{bmatrix} 0.40\hat{8} \\ 0.817\hat{\imath} \\ 0.408\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 1^v \\ i^v \\ \hat{\jmath} \end{bmatrix}) \times \begin{bmatrix} 0.40\hat{8} \\ 0.817\hat{\imath} \\ 0.408\hat{\jmath} \end{bmatrix}$$

$$v_{rot} =^{-} 0.817 \begin{bmatrix} 0.40\hat{8} \\ 0.817\hat{\imath} \\ 0.408\hat{\jmath} \end{bmatrix} + 0.866(\begin{bmatrix} 1.22\hat{5} \\ 0.816i^v \\ 0.409\hat{\jmath} \end{bmatrix}) +^{-} 0.5(\begin{bmatrix} 1.22\hat{5} \\ 0.816i^v \\ 0.409\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 0.40\hat{8} \\ 0.817\hat{\imath} \\ 0.408\hat{\jmath} \end{bmatrix})$$

$$v_{rot} = \begin{bmatrix} 0.333^v \\ 0.667i^v \\ 0.333j^v \end{bmatrix} + \begin{bmatrix} 1.06\hat{1} \\ 0.707i^v \\ 0.354\hat{\jmath} \end{bmatrix} + \begin{bmatrix} 0.33\hat{1} \\ 0.3167\hat{\imath} \\ 0.667j^v \end{bmatrix}$$

$$v_{rot} = \begin{bmatrix} 1.06\hat{1} \\ 1.207i^v \\ 0.646j^v \end{bmatrix}$$

Rotation 2:

$$v_{rot} = (\begin{bmatrix} 0.26\hat{7} \\ 0.535\hat{\imath} \\ 0.802\hat{\jmath} \end{bmatrix} \cdot \begin{bmatrix} 1.06\hat{1} \\ 1.207i^v \\ 0.646j^v \end{bmatrix}) \begin{bmatrix} 0.26\hat{7} \\ 0.535\hat{\imath} \\ 0.802\hat{\jmath} \end{bmatrix} + sin45^{\hat{o}}(\begin{bmatrix} 0.26\hat{7} \\ 0.535\hat{\imath} \\ 0.802\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 1.06\hat{1} \\ 1.207i^v \\ 0.646j^v \end{bmatrix}) + cos45^{\hat{o}}((\begin{bmatrix} 0.26\hat{7} \\ 0.535\hat{\imath} \\ 0.802\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 1.06\hat{1} \\ 1.207i^v \\ 0.646j^v \end{bmatrix}) \times \begin{bmatrix} 0.26\hat{7} \\ 0.535\hat{\imath} \\ 0.802\hat{\jmath} \end{bmatrix})$$

$$v_{rot} =^{-} 0.880 \begin{bmatrix} 0.26\hat{7} \\ 0.535\hat{\imath} \\ 0.802\hat{\jmath} \end{bmatrix} + 0.707(\begin{bmatrix} 0.62\hat{2} \\ 1.023\hat{\imath} \\ 0.89j^v \end{bmatrix}) + 0.707(\begin{bmatrix} 0.62\hat{2} \\ 1.023\hat{\imath} \\ 0.89j^v \end{bmatrix} \times \begin{bmatrix} 0.26\hat{7} \\ 0.535\hat{\imath} \\ 0.802\hat{\jmath} \end{bmatrix})$$

$$v_{rot} = \begin{bmatrix} 0.235^v \\ 0.471i^v \\ 0.706j^v \end{bmatrix} + \begin{bmatrix} 0.4\hat{4} \\ 0.724\hat{\imath} \\ 0.629j^v \end{bmatrix} + \begin{bmatrix} 0.91\hat{7} \\ 0.521i^v \\ 0.043\hat{\jmath} \end{bmatrix}$$

$$v_{rot} = \begin{bmatrix} 1.12\hat{2} \\ 0.268i^v \\ 1.292j^v \end{bmatrix}$$

Rotation 3:

$$v_{rot} =$$
$$(\begin{bmatrix} 0.456\hat{} \\ 0.57\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} \cdot \begin{bmatrix} 1.122\hat{} \\ 0.268i^{v} \\ 1.292j^{v} \end{bmatrix}) \begin{bmatrix} 0.456\hat{} \\ 0.57\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} + sin60\hat{}^{o}(\begin{bmatrix} 0.456\hat{} \\ 0.57\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 1.122\hat{} \\ 0.268i^{v} \\ 1.292j^{v} \end{bmatrix}) + cos60\hat{}^{o}((\begin{bmatrix} 0.456\hat{} \\ 0.57\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} \times \begin{bmatrix} 1.122\hat{} \\ 0.268i^{v} \\ 1.292j^{v} \end{bmatrix}) \times \begin{bmatrix} 0.456\hat{} \\ 0.57\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix})$$

$$v_{rot} =^{-} 0.525 \begin{bmatrix} 0.456\hat{} \\ 0.57\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix} + 0.866(\begin{bmatrix} 0.553^{v} \\ 1.357\hat{\imath} \\ 0.762j^{v} \end{bmatrix}) + 0.5(\begin{bmatrix} 0.553^{v} \\ 1.357\hat{\imath} \\ 0.762j^{v} \end{bmatrix} \times \begin{bmatrix} 0.456\hat{} \\ 0.57\hat{\imath} \\ 0.684\hat{\jmath} \end{bmatrix})$$

$$v_{rot} = \begin{bmatrix} 0.239^{v} \\ 0.299i^{v} \\ 0.359j^{v} \end{bmatrix} + \begin{bmatrix} 0.479^{v} \\ 1.175\hat{\imath} \\ 0.66j^{v} \end{bmatrix} + \begin{bmatrix} 0.681\hat{} \\ 0.015\hat{\imath} \\ 0.467j^{v} \end{bmatrix}$$

$$v_{rot} = \begin{bmatrix} 0.037^{v} \\ 0.891\hat{\imath} \\ 1.485j^{v} \end{bmatrix}$$

Equivalent $\mathbb{R}3$ ***Rotation***

Using the Vector Rotation functionality of the Omni Quaternion Calculator[13] for an equivalent chain of rotations in $\mathbb{R}3$, results as follows:

- Rotation 1: $q_v(-1, -1, 1)$ rotated by 120^o around $(1, 2, 1)$ moves to $(1.061, -1.207, -0.646)$
- Rotation 2: $(1.061, -1.207, -0.646)$ rotated by 45^o around $(1, 2, 3)$ moves to $(1.122, -0.268, -1.292)$
- Rotation 3: $(1.122, -0.268, -1.292)$ rotated by 60^o around $(4, 5, 6)$ moves to $(-0.037, 0.891, -1.485)$

This matches the Wave Number rotation result obtained above.

Equivalent Quaternion Rotation

Setting up Quaternion Equations. The following describes the quaternion equivalent of this chain of Wave Number rotations:

- Rotation 1: Rotate $(-i, -j, k)$ by 120^o around $(i, 2j, k)$
- Rotation 2: Rotate the result of rotation 1 by 45^o around $(i, 2j, 3k)$
- Rotation 3: Rotate the result of rotation 2 by 60^o around $(4i, 5j, 6k)$

The unit vectors of the axes of rotation are:

- Rotation 1: $(i + 2j + k)/\sqrt{(1^2 + 2^2 + 1^2} = (i + 2j + k)/2.445 = (0.408i + 0.816j + 0.408k)$
- Rotation 2: $(i + 2j + 3k)/\sqrt{(1^2 + 2^2 + 3^2)} = (i + 2j + 3k)/3.742 = (0.267i + 0.535j + 0.802k)$
- Rotation 3: $(4i + 5j + 6k)/\sqrt{(4^2 + 5^2 + 6^2)} = (4i + 5j + 6k)/8.774 = (0.456i + 0.57j + 0.684k)$

The quaternions of rotations are:

- Rotation 1: $q_{r1} = 0.5 + (0.408i + 0.816j + 0.408k) * 0.866 = (0.5 + 0.354i + 0.707j + 0.354k)$
- Rotation 2: $q_{r2} = 0.924 + (0.267i + 0.535j + 0.802k) * 0.383 = (0.924 + 0.102i + 0.205j + 0.307k)$
- Rotation 3: $q_{r3} = 0.866 + (0.456i + 0.57j + 0.684k) * 0.5 = (0.866 + 0.228i + 0.285j + 0.342k)$

Performing Quaternion Equations. The quaternion of rotation, q_{r1} rotates the quaternion vector, q_v as follows:

- $(0.5 + 0.354i + 0.707j + 0.354k)(-i - j + k) = (0.707 + 0.561i - 1.207j + 0.854k)$

  q_{r2} rotates the result of the first rotation:

- $(0.924 + 0.102i + 0.205j + 0.307k)(0.707 + 0.561i - 1.207j + 0.854k) = (0.581 + 1.135i - 0.886j + 0.767k)$

  q_{r3} rotates the result of the second rotation:

- $(0.866 + 0.228i + 0.285j + 0.342k)(0.581 + 1.135i - 0.886j + 0.767k) = (0.234 + 1.637i - 0.388j + 0.338k)$

Completing the Quaternion Sandwich. The quaternion sandwich needs to be completed to obtain a result in $\mathbb{R}3$. Consequently, the result so far needs to be multiplied by the conjugate of the three rotations $(q_{r3} * q_{r2}) * q_{r1}$.

- The product of the three rotations is $(q_{r3} * q_{r2}) * q_{r1} = (0.866 + 0.228i + 0.285j + 0.342k)(0.924 + 0.102i + 0.205j + 0.307k)(0.5 + 0.354i + 0.707j + 0.354k) = (-0.304 + 0.095i + 0.736j + 0.597k)$
- Therefore the conjugate of the three rotations is $(-0.304 - 0.095i - 0.736j - 0.597k)$
- Giving the product of the result so far and the conjugate as $(0.234 + 1.637i - 0.388j + 0.338k)(-0.304 - 0.095i - 0.736j - 0.597k) = (-0.039i + 0.891j - 1.485k)$

The result matches the Wave Numbers outcome of $(0.037^{v} + 0.891\hat{\imath} + 1.485j^{v})$, allowing for minor rounding differences due to the number of multiplications.

6.3 Rotication

6.3.1 What is Rotication in R3

The rotication operation is grounded in the principles outlined in its axioms. Rotication consists of two steps. The first step is identical to rotation, resulting in a circular movement of a point.

The second step takes the result of the rotation and multiplies it by the magnitude of the axis of rotation, leading to a linear movement of the point. Thus, the new point remains on the same line from the origin as that of a pure rotation.

Typically, one of the pair of Opposite Values defining the axis is the origin (0, 0, 0), but any pair of Opposite Values can define an axis.

6.3.2 Syntax

Rotication requires an amount of rotation, an axis, and a starting point. Rotication is expressed as follows, where $^?$ represents the axis around which the rotication takes place. The default axis is formed by $(0,0,0)$ and $(0,0,\hat{j})$:

(Amount of rotication) $R \circlearrowleft^?$ (Opposite Value).

In sample calculations, formulae, and the calculator, the axis is assumed to be formed by the origin and the given Opposite Value. If an axis does not pass through the origin, it needs to be translated to the origin, the rotation performed, and then the result translated back by reversing the original translation.

6.3.3 Formula

R3 rotication is calculated using the magnitude of the axis, $\sqrt{(|x_a|^2 + |y_a|^2 + |z_a|^2)}$ and the Wave Number formula as follows:

$$v_{Rotication} = \sqrt{(|x_a|^2 + |y_a|^2 + |z_a|^2)} * ((u_r \cdot v)u_r + sin\theta(u_r \times v) + cos\theta((u_r \times v) \times u_r))$$

where x_a, y_a and z_a are the coordinates of the point that forms the axis with the origin.

6.3.4 Examples

Rotication where magnitude of the axis is 1

$\frac{\hat{\pi 2}}{R} \circlearrowleft (\hat{1}, 0, 0)$

$= \sqrt{(1^2 + 0^2 + 0^2)}(((0,0,\hat{j}) \cdot (\hat{1},0,0)) * (0,0,\hat{j}) + 1 * ((0,0,\hat{j}) \times (\hat{1},0,0)) + 0 * ((0,0,\hat{j}) \times (\hat{1},0,0)) \times (0,0,\hat{j}))$

$= 1 * (0 * (0,0,\hat{j})) + 1 * (\hat{i}) + 0$
$= \hat{i}$

Rotication where magnitude of the axis is > 1

$\frac{2\hat{\pi}}{3} R \circlearrowleft^{(\hat{,}2\hat{i},\hat{j})} (1^v + i^v + \hat{j})$

$= \sqrt{(1^2 + 2^2 + 1^2)}((ur \cdot v) * ur) + sin2/3\hat{\pi}(ur \times v) + cos2/3\hat{\pi} * ((ur \times v) \times ur))$
$= 2.449 * ((0.40\hat{8}, 0.816\hat{i}, 0.408\hat{j}) \cdot (1^v, i^v, \hat{j})) * (0.40\hat{8}, 0.816\hat{i}, 0.408\hat{j})$
$+ 0.866 * ((0.40\hat{8}, 0.816\hat{i}, 0.408\hat{j}) \times (1^v, i^v, \hat{j}))$
$+ ^- 0.5 * ((0.40\hat{8}, 0.816\hat{i}, 0.408\hat{j}) \times (1^v, 1i^v, j^v)) \times (0.40\hat{8}, 0.816\hat{i}, 0.408\hat{j}))$
$= 2.449 * (1.06\hat{1} + 1.207i^v + 0.646j^v) = (2.\hat{6} + 2.96i^v + 1.58j^v)$

6.4 Multiplication

6.4.1 Introduction

Multiplication in R3 is a scalar operation that incorporates rotation through Opposite Types and Signs. It follows a "times-add" process, with the Opposite Sign of the result determined by the multiplication table.

The Orthogonal Rule states that multiplication by a unitary is equivalent to rotation around the unitary's axis by 90^o. This principle enabled the development of the multiplication table below directly from the rotations of a 3d sphere.

Table 2 R3 rotation/multiplication table

*	$1^\wedge$	1^v	$i^\wedge$	i^v	$j^\wedge$	j^v
$1^\wedge$	$1^\wedge$	1^v	$j^\wedge$	j^v	i^v	$i^\wedge$
1^v	$1^\wedge$	1^v	j^v	$j^\wedge$	$i^\wedge$	i^v
$i^\wedge$	j^v	$j^\wedge$	$i^\wedge$	i^v	$1^\wedge$	1^v
i^v	$j^\wedge$	j^v	$i^\wedge$	i^v	1^v	$1^\wedge$
$j^\wedge$	$i^\wedge$	i^v	1^v	$1^\wedge$	$j^\wedge$	j^v
j^v	i^v	$i^\wedge$	$1^\wedge$	1^v	$j^\wedge$	j^v

6.4.2 Syntax

The Operator is defined as the first term in a multiplication, which acts on the Operand, the second term of the multiplication. The symbol * represents the operation. For example:

$$3i^v * 4j^\wedge = 12^v$$

Here, $3i^v$ is the operator and $4j^\wedge$ is the operand. The operator can be an Opposite Value or a Counter, while the operand can only be an Opposite Value. For example:

$$4 * 3i^\wedge = 12i^\wedge$$

Assume multiplication when an Opposite Value precedes or follows a round bracket, or when two round brackets are together. For example:

$$4(5^\wedge + 2i^v) = 20^\wedge + 8i^v$$

$$(5i^\wedge + 2^v)4j^v = 20^v + 8i^v$$

$$(5^\wedge + 2i^v)(4i^v + 7j^\wedge) = 14^v + 43i^v + 20j^v$$

Counters cannot be multiplied together on their own in an operation, as the result of any operation must be an Opposite Value. However, they can be multiplied with terms.

6.4.3 Examples

Multiplication is not commutative in R3:

$$
\begin{aligned}
&3i^v * \hat{4} = 12\hat{\jmath} \\
&\hat{4} * 3i^v = 12j^v \\
&(5i^v + 2\hat{\jmath})(6\hat{\imath} + 3^v) = (5i^v * 6\hat{\imath} + 5i^v * 3^v + 2\hat{\jmath} * 6\hat{\imath} + 2\hat{\jmath} * 3^v) \\
&= 30\hat{\imath} + 15j^v + 1\hat{2} + 6i^v \\
&= 12^v + 24\hat{\imath} + 15j^v \\
&(6\hat{\imath} + 3^v)(5i^v + 2\hat{\jmath}) = (6\hat{\imath} * 5i^v + 6\hat{\imath} * 2\hat{\jmath} + 3^v * 5i^v + 3^v * 2\hat{\jmath}) \\
&= 30i^v + 1\hat{2} + 15\hat{\jmath} + 6\hat{\imath} \\
&= 1\hat{2} + 24i^v + 15\hat{\jmath}
\end{aligned}
$$

Multiplication is not associative in R3:

$$
\begin{aligned}
&(\hat{\jmath} * \hat{\jmath}) * 1^v = \hat{\jmath} * 1^v = i^v \\
&\hat{\jmath} * (\hat{\jmath} * 1^v) = \hat{\jmath} * i^v = \hat{1} \\
&(2i^v * 4\hat{\jmath}) * \hat{3} = 8^v * \hat{3} = 2\hat{4} \\
&2i^v * (4\hat{\jmath} * \hat{3}) = 2i^v * 12\hat{\imath} = 24\hat{\imath}
\end{aligned}
$$

Multiplication is distributive as long as all Opposite Values are consolidated. Note a consolidated Opposite Value means that any Opposite Values of the same Opposite Type have been added together.

$$
\begin{aligned}
&(5\hat{\jmath} + 4i^v)(\hat{6} + 3i^v) \\
&= 5\hat{\jmath} * \hat{6} + 5\hat{\jmath} * 3i^v + 4i^v * \hat{6} + 4i^v * 3i^v \\
&= 30\hat{\imath} + 1\hat{5} + 24\hat{\jmath} + 12i^v \\
&= 1\hat{5} + 18\hat{\imath} + 24\hat{\jmath}
\end{aligned}
$$

$$(2^v + 1\hat{0})(9^v + 4\hat{4}) = 8^v * 3\hat{5} = 28\hat{0}$$

but

$$2^v * 9^v + 2^v * 4\hat{4} + 1\hat{0} * 9^v + 1\hat{0} * 4\hat{4} = 18^v + 8\hat{8} + 90^v + 44\hat{0} = 420^v$$

$$(\hat{1} + \hat{\imath} + \hat{\jmath}) * \hat{1} = (\hat{1} + j^v + \hat{\imath})$$

$$
\begin{aligned}
&(1^{\wedge} + i^{\wedge} + j^{\wedge})(1^{\wedge} + j^{v} + i^{\wedge}) \\
&= (1^{\wedge} + j^{\wedge} + i^{\wedge}) + (j^{v} + 1^{v} + i^{\wedge}) + (i^{\wedge} + j^{v} + 1^{v}) \\
&= (1^{v} + 3i^{\wedge} + j^{v})
\end{aligned}
$$

$$
\begin{aligned}
&(6^{\wedge} +^{-} 4^{\wedge})(2^{v} + 4i^{v} + 3j^{v}) = 2^{\wedge}(2^{v} + 4i^{v} + 3j^{v}) \\
&= 4^{v} + 6i^{\wedge} + 8j^{v}
\end{aligned}
$$

Counters cannot be multiplied with only themselves, as the result of any calculation must be an Opposite Value. Therefore, $3 * 5 * 2j^{\wedge} = 30j^{\wedge}$ is valid, but $3 * 5 = 15$ is not permitted since 15 is not an Opposite Value.

$$5(6^{\wedge} + 3i^{v} + 4j^{\wedge}) = 30^{\wedge} + 15i^{v} + 20j^{\wedge}$$

A flip sign in front of a counter indicates that the Opposite Sign should be changed to the other Opposite Sign after multiplication. Note that two flips cancel out.

$$
\begin{aligned}
&{}^{-}5(6^{\wedge} + 3i^{v} + 4j^{\wedge}) = (30^{v} + 15i^{v} + 20j^{v}) \\
&{}^{-}2 *^{-} 2 * 2j^{\wedge} = 8j^{\wedge}
\end{aligned}
$$

6.4.4 Multiplication Algebraic Formula

It can be seen from the multiplication table that the product of each pair of Opposite Value Types and Signs results in one Opposite Type and Sign.

This allows the multiplication calculation to be written in terms of Opposite Value types as in the following formulae for the multiplication of the two points a, with coordinates (x_a, y_a, z_a), and b, with coordinates (x_o, y_o, z_o):

${}^{\wedge}/{}^{v}$ value: $(x_a * x_o + y_a * z_o + z_a * y_o)$
$i^{\wedge}/i^{v}$value: $(y_a * y_o + x_a * z_o + z_a * x_o)$
$j^{\wedge}/j^{v}$ value: $(z_a * z_o + x_a * y_o + y_a * x_o)$

6.4.5 Multiplication Geometric Formula

An alternative method to using the multiplication table for calculating multiplication is to use the following trigonometric multiplication formula.

The distance R of any point (x, y, z) from the origin as given by Pythagoras' theorem is $R = \sqrt{(|x|^2 + |y|^2 + |z|^2)}$.

Any point (x, y, z) can be defined in terms of the angles $\psi x, \psi y, \psi z$.

ψx is the angle formed between the line formed by (x, y, z) and the origin $(0, 0, 0)$ and the x-axis represented by the unit line $(1^{\wedge}, 0, 0)$. ψy is the angle formed between

the line formed by (x, y, z) and the origin $(0, 0, 0)$ and the y-axis represented by the unit line $(0, \hat{\imath}, 0)$. ψz is the angle formed between the line formed by (x, y, z) and the origin $(0, 0, 0)$ and the z-axis represented by the unit line $(0, 0, \hat{\jmath})$.
Given that $(Ax, Ay, Az) * (Bx, By, Bz) = (Cx, Cy, Cz)$, the formula for calculating (Cx, Cy, Cz) is:

$$Cx = RA * RB(cos(\psi Ax)\hat{}.cos(\psi Bx)\hat{} + cos(\psi Ay)\hat{\imath}.cos(\psi Bz)\hat{\jmath} + cos(\psi Az)\hat{\jmath}.cos(\psi By)\hat{\imath})$$

$$Cy = RA * RB(cos(\psi Ax)\hat{}.cos(\psi Bz)\hat{\jmath} + cos(\psi Ay)\hat{\imath}.cos(\psi By)\hat{\imath} + cos(\psi Az)\hat{\jmath}.cos(\psi Bx)\hat{})$$

$$Cz = RA * RB(cos(\psi Ax)\hat{}.cos(\psi By)\hat{\imath} + cos(\psi Ay)\hat{\imath}.cos(\psi Bx)\hat{} + cos(\psi Az)\hat{\jmath}.cos(\psi Bz)\hat{\jmath})$$

Example

Using the multiplication table :

$$(6.3^v + 4.7\hat{\imath} + 5.3\hat{\jmath}) * (3.9\hat{} + 7.5i^v + 2.4j^v) = (53.04\hat{} + 29.7i^v + 16.2\hat{\jmath})$$

Using the multiplication formula:

$$RA = \sqrt{(|6.3|^2 + |4.7|^2 + |5.3^2)} = 9.48.$$
$$RB = (\sqrt{(|3.9|^2 + |7.5|^{2}\hat{} + |2.4^2)} = 8.79.$$

$$cos(\psi Ax)\hat{} = (^{-}6.3/9.48)\hat{} = 0.665^v$$
$$cos(\psi Ay)\hat{\imath} = (4.7/9.48)\hat{\imath} = 0.496\hat{\imath}$$
$$cos(\psi Az)\hat{\jmath} = (5.3/9.48)\hat{\jmath} = 0.559\hat{\jmath}$$
$$cos(\psi Bx)\hat{} = (3.9/8.79)\hat{} = 0.444\hat{}$$
$$cos(\psi By)\hat{\imath} = (^{-}7.5/8.79)\hat{\imath} = 0.853i^v$$
$$cos(\psi Bz)\hat{\jmath} = (^{-}2.4/8.79)\hat{\jmath} = 0.273j^v$$

$$\begin{aligned} Cx &= 9.48 * 8.79(0.665^v * 0.444\hat{} + 0.496\hat{\imath} * 0.273j^v + 0.559\hat{\jmath} * 0.853i^v) \\ &= 83.31 * (0.295\hat{} + 0.135^v + 0.477\hat{}) \\ &= 83.31 * 0.6368\hat{} \\ &= 53.04\hat{} \end{aligned}$$

$$\begin{aligned} Cy &= 9.48 * 8.79(0.665^v * 0.273j^v + 0.496\hat{\imath} * 0.853i^v + 0.559\hat{\jmath} * 0.444\hat{}) \\ &= 83.31 * (0.182i^v + 0.423i^v + 0.248\hat{\imath}) \\ &= 83.31 * 0.357\hat{} \\ &= 29.7i^v \end{aligned}$$

$$Cz = 9.48 * 8.79(0.665^v * 0.853i^v + 0.496\hat{\imath} * 0.444\hat{} + 0.559\hat{\jmath} * 0.273j^v)$$

$$= 83.31 * (0.567\hat{\jmath} + 0.22j^v + 0.153j^v)$$
$$= 83.31 * 0.19\hat{4}$$
$$= 16.2\hat{\jmath}$$

The formula calculates (Cx, Cy, Cz) as $(53.0\hat{4} + 29.7i^v + 16.2\hat{\jmath})$, the same as when using the multiplication table.

6.5 Dot Product

6.5.1 Introduction

Dot product multiplication in R3 uses the same approach as in classical mathematics. [9]

The dot product of the points $a(a_x, a_y, a_z)$ and $b(b_x, b_y, b_z)$ is written as $a \cdot b$ and is given by the following formula, where a flip $(^-)$ precedes the Counters when the Opposite Value is $^v, i^v$ or j^v:

$$a \cdot b = (^?|a_x| *^? |b_x|) + (^?|a_y| *^? |b_y|) + (^?|a_z| *^? |b_z|)$$

It can also be defined geometrically as the product of the magnitudes of the points and the cosine of the angle (θ) between them. This relationship is expressed as:

$$x \cdot y = |x||y|cos\phi$$

6.5.2 Examples

$$\hat{\jmath} \cdot \hat{} = 0$$
$$\hat{\jmath} \cdot \hat{\jmath} = 1$$
$$\hat{\jmath} \cdot (^v + i^v + \hat{\jmath}) = 1$$
$$2^v \cdot \hat{2} =^- 2 * 2 =^- 4$$
$$2i^v \cdot 4i^v =^- 2 *^- 4 = 8$$
$$(0.577^v + 0.577i^v + 0.577\hat{\jmath}) \cdot \hat{\jmath} = 0.577$$
$$(\hat{4} + 5\hat{\imath} + 6\hat{\jmath}) \cdot (\hat{1} + 2\hat{\imath} + 3\hat{\jmath}) = 32$$
$$(4^v + 5i^v + 6\hat{\jmath}) \cdot (1^v + 2\hat{\imath} + 3j^v) =^- 24$$
$$(6.3^v + 4.7\hat{\imath} + 5.3\hat{\jmath}) \cdot (3.\hat{9} + 7.5i^v + 2.4j^v)$$
$$=^- 6.3 * 3.9 + 4.7 *^- 7.5 + 5.3 *^- 2.4$$
$$=^- 24.57 +^- 35.25 +^- 12.72$$
$$=^- 72.54$$

6.6 Cross Product

6.6.1 Introduction

Cross product multiplication in R3 uses the same approach as in classical mathematics. [9]

The cross product of the points $a(a_x, a_y, a_z)$ and $b(b_x, b_y, b_z)$ is written as $a \times b$ and is given by the following formula:

$$a \times b = ((a_y * b_z + a_z * b_y), (a_z * b_x + a_x * b_z), (a_x * b_y + a_y * b_x))$$

6.6.2 Examples

$$\begin{aligned}
&(4^v + 5i^v + 6\hat{\jmath}) \times (1^v + 2\hat{\imath} + 3j^v) = \\
&= ((5i^v * 3j^v + 6\hat{\jmath} * 2\hat{\imath}), (6\hat{\jmath} * 1^v + 4^v * 3j^v), (4^v * 2\hat{\imath} + 5i^v * 1^v)) \\
&= ((15^v + 1\hat{2}), (6i^v + 12i^v), (8j^v + 6j^v)) \\
&= (3^v, 6i^v, 2\hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
&(6.3^v + 4.7\hat{\imath} + 5.3\hat{\jmath}) \times (3.\hat{9} + 7.5i^v + 2.4j^v) \\
&= ((4.7\hat{\imath} * 2.4j^v + 5.3\hat{\jmath} * 7.5i^v), (5.3\hat{\jmath} * 3.\hat{9} + 6.3^v * 2.4j^v), (6.3^v * 7.5i^v + 4.7\hat{\imath} * 3.\hat{9})) \\
&= ((11.28^v + 39.7\hat{5}), (20.67\hat{\imath} + 15.12i^v), (47.25\hat{\jmath} + 18.33j^v)) \\
&= (28.4\hat{7}, 5.55\hat{\imath}, 28.92\hat{\jmath})
\end{aligned}$$

6.7 Division

6.7.1 Introduction

In the context of R3, division is defined as the inverse operation of multiplication. The result of dividing an operand by an operator is an expression such that when the operator is multiplied by this result, it yields the original operand.

R3 division is a scalar operation that incorporates rotation through Opposite Types and Signs. As the inverse of multiplication, it necessitates reversing both the circular motion of multiplication through the plane and the linear motion along a number line.

The circular motion's inverse is determined using the R3 division table provided below.

The scalar part of the operation inverse is determined similarly to division in classical mathematics.

The division table is derived very simply from the multiplication table. For example:

$$\hat{1} * i^v = j^v \implies i^v = j^v / \hat{1}$$

Table 3 R3 Division table

$/$	$1^\wedge$	1^v	$i^\wedge$	i^v	$j^\wedge$	j^v
$1^\wedge$	$1^\wedge$	$^\wedge$	$j^\wedge$	j^v	i^v	$i^\wedge$
1^v	1^v	1^v	j^v	$j^\wedge$	$i^\wedge$	i^v
$i^\wedge$	j^v	$j^\wedge$	$i^\wedge$	$i^\wedge$	$1^\wedge$	1^v
i^v	$j^\wedge$	j^v	i^v	i^v	1^v	$1^\wedge$
$j^\wedge$	$i^\wedge$	i^v	1^v	$1^\wedge$	$j^\wedge$	$j^\wedge$
j^v	i^v	$i^\wedge$	$1^\wedge$	1^v	j^v	j^v

6.7.2 Syntax

In R3 division, the divisor acts as the Operator, and the numerator serves as the Operand. The symbol / represents the operation. The result derives its Opposite Sign from the division table.

In the equation, $12^\wedge/4i^v = 3j^v, 4i^v$ is the Operator and $12^\wedge$ is the Operand and $1^\wedge/i^v$ in the division table gives a result of j^v. The Operator can be an Opposite Value or a Counter, while the Operand can only be an Opposite Value.

As in classical mathematics, you cannot divide by zero. So, equations such as $1^\wedge/0$ and $j^v/0$ are undefined.

Division is not commutative:

$$6j^\wedge/3i^v = 2^\wedge, 3i^v/6j^\wedge = 0.5^v$$

Division is not associative:

$$(24j^\wedge/8i^v)/2^v = 3^\wedge/2^v = 3/2^\wedge$$
$$24^\wedge/(8i^v/2^v) = 24^\wedge/4j^v = 6i^\wedge$$

A Counter cannot be divided directly by another Counter in a standalone calculation because the result of such a division must be an Opposite Value. However, it is possible to divide terms by Counters.

When dividing by Counters, the operation is performed on the individual Opposite Value elements. For example:

$$(6^\wedge + 9i^v + 12j^\wedge)/3 = (2^\wedge + 3i^v + 4j^\wedge)$$

6.7.3 Use of Simultaneous Equations

According to the theorem on multiplicative inverses, R3 lacks multiplicative inverses. Therefore, the Wave Number system employs an alternative method to compute the result of division in R3, using simultaneous equations.

Consider the following equation:

$$(3^v+4i^\wedge+5j^v)(2^\wedge+3i^\wedge+4j^\wedge) = (7+10i^v+21j^\wedge) \implies (7+10i^v+21j^\wedge)/(3^v+4i^\wedge+5j^v) = (2^\wedge+3i^v+4j^\wedge)$$

To obtain the result of the division, you need to solve the following equation:

$$(\hat{7} + 10i^v + 21\hat{\jmath})/(3^v + 4\hat{\imath} + 5j^v) = (x? + y? + z?)$$

The following method involves using simultaneous equations to determine the result of dividing R3.

Step 1: Identifying the R3 Simultaneous Equations Needed for Division
Rewrite the equation above as:

$$(3^v + 4\hat{\imath} + 5j^v) * (x + y + z) = (7 + 10i^v + 21\hat{\jmath})$$

The expression $(3^v + 4\hat{\imath} + 5j^v)$ serves as the Operator of the multiplication and is represented by (x_a, y_a, z_a). Each element corresponds to an axis of one of the three rotations of the multiplication. The Operand of the division is $(\hat{7} + 10i^v + 21\hat{\jmath})$ and is represented by (x_0, y_0, z_0).

The formulae for calculating multiplication are:

$\hat{}/^v$ value: $(x_a * x_o + y_a * z_o + z_a * y_o)$
$\hat{\imath}/i^v$value: $(y_a * y_o + x_a * z_o + z_a * x_o)$
$\hat{\jmath}/j^v$ value: $(z_a * z_o + x_a * y_o + y_a * x_o)$

Accordingly, this gives three equations:

A: $3^v * x + 4\hat{\imath} * z + 5j^v * y = \hat{7}$
B: $3^v * z + 4\hat{\imath} * y + 5j^v * x = 10i^v$
C: $3^v * y + 4\hat{\imath} * x + 5j^v * z = 21\hat{\jmath}$

Step 2: Transformation into Simultaneous Equations
Transform Equation A into a simultaneous equation as follows:

$3^v * x + 4\hat{\imath} * z + 5j^v * y = \hat{7}$
$\implies 3x + 4z + 5y = 7$
$\implies x = (7 +^- 4z +^- 5y)/3$

The multiplication table indicates that the product of Opposite Values of a ^{v} with a $\hat{}$ results in a $\hat{}/^v$ value that retains the sign of the operand. Thus, $3^v * x$ becomes $3x$.
Similarly $4\hat{\imath}*z$ produces a $\hat{}/^v$ value with the same sign as in z. Thus, $4i$ becomes $4z$.
Likewise, $5j^v * y$ results in a $\hat{}/^v$ value with the same sign as in y. Thus, $5j^v * y$ becomes $5y$.

The Opposite Sign of the result is ˆ and so the result becomes 7.

The Opposite Types and Signs have been eliminated, yielding equation A-modified:

$$3x + 4z + 5y = 7$$

$$x = (\hat{7} +^- 4z +^- 5y)/3$$

Transform Equation B into a simultaneous equation as follows:

$3^v * z + 4\hat{\imath} * y + 5j^v * x = 10i^v$
$\implies 3z + 4y +^- 5x =^- 10$
$\implies z = (^-10 +^- 4y + 5x)/3$

The product of a ^{v} and a j^v or $\hat{\jmath}$ result in an $\hat{\imath}/i^v$ that retains the sign of the operand. So, $3^v * z$ becomes $3z$.

Similarly $4\hat{\imath} * y$ becomes $4y$ with the same sign as in y.

However, the product of a j^v and an x results in a sign reversal, thus $5j^v * x$ becomes ^-5x.

The Opposite Sign of the result is ^{v} and so the result becomes $^-10$.

The Opposite Signs have been eliminated yielding the equation B-modifed:

$$\implies 3z + 4y +^- 5x =^- 10$$
$$\implies z = (^-10 +^- 4y + 5x)/3$$

Finally, equation C generates the following simultaneous equation C-modified:

$$^-3y +^- 4x + 5z = 21$$
$$\implies y = (^-21 +^- 4x + 5z)/3$$

Step 3: Solving the Simultaneous Equations

Substituting for $y = (^-21 +^- 4x + 5z)/3$ from Equation C-modified into Equation A-modified above gives:
$3x + 4z + (5/3)(^-21 +^- 4x + 5z) = 7$
$\implies 9x + 12z +^- 105 +^- 20x + 25z = 21$
$\implies {}^-11x + 37z = 21 + 105 = 126$ giving Equation D

Secondly, a similar substitution for y in Equation B-modified above gives:
$3z + (4/3)(^-21 +^- 4x + 5z) +^- 5x =^- 10$
$\implies 9z +^- 84 +^- 16x + 20z +^- 15x =^- 30$
$\implies 29z +^- 31x = 54 => z = (54 + 31x)/29$ giving Equation E

Thirdly, substituting for $z = (54+31x)/29$ from Equation E into Equation D gives:

$^{-}11x + 37(54 + 31x)/29 = 126$
$\implies {}^{-}319x + 1998 + 1147x = 3654$
$\implies 828x = 1656 => x = 2$

Next, substituting for x with 2 in Equation D gives:
$^{-}11 * 2 + 37z = 126$
$\implies 37z = 126 + 22$
$\implies z = 148/37 = 4$

Finally, substituting for x with 2 and z with 4 in Equation A-modifed gives:

$3 * 2 + 4 * 4 + 5y = 7$
$\implies 5y = 7 +^{-} 22 =^{-} 15$
$\implies y =^{-} 3$

The variable x represents the $\hat{}/^{v}$ Counter and is 2 and so gives a solution for x of $\hat{2}$. The variable y represents the $\hat{i}/i^{v}$ Counter and is $^{-}3$ and so gives a solution for y of $3i^{v}$. The variable z represents the $\hat{j}/j^{v}$ Counter and is 4 and so gives a solution for z of $4\hat{j}$. So, the overall solution is $(\hat{2} + 3i^{v} + 4\hat{j})$.

7 Results and Discussion

All the calculations in the previous section have been verified with their equivalent calculations in $\mathbb{R}3$ and quaternions using the Vector Rotation and Quaternion functionality of the Omni Quaternion Calculator[13].

7.1 Rotation

7.1.1 Comparison of Wave Number and Quaternion Rotation Tables.

The following tables represents the quaternion and Wave Number multiplication rules. The leftmost column indicates the multiplier, while the topmost row represents the multiplicand.

To interpret the quaternion table, consider $i * j = k$. This means that multiplying j by i results in k. Here, i denotes a rotation around the x-axis. Similarly, $j * i = -k$ signifies that multiplying i by j yields $-k$. Rotating i, j or k by themselves results in -1, which is a scalar. Finally, multiplying the scalar 1 by 1 remains 1.

Table 4 Quaternion multiplication table

$*$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

The entries in the rows of the R3 multiplication table can be directly compared to those in the quaternion table.

In the R3 system, there is no equivalent of the scalar 1. Instead, $\hat{1}, \hat{\imath}$ and $\hat{\jmath}$ correspond to the quaternion values i, j and k respectively. Row i of the quaternion table is compared with its equivalent Wave Number, row $\hat{1}$. Quaternion values such as $i * -i$ are calculated as the result of $i * i * -1$.

Table 5 R3 rotation/multiplication table

$*$	$\hat{1}$	1^v	$\hat{\imath}$	i^v	$\hat{\jmath}$	j^v
$\hat{1}$	$\hat{1}$	1^v	$\hat{\jmath}$	j^v	i^v	$\hat{\imath}$
1^v	$\hat{1}$	1^v	j^v	$\hat{\jmath}$	$\hat{\imath}$	i^v
$\hat{\imath}$	j^v	$\hat{\jmath}$	$\hat{\imath}$	i^v	$\hat{1}$	1^v
i^v	$\hat{\jmath}$	j^v	$\hat{\imath}$	i^v	1^v	$\hat{1}$
$\hat{\jmath}$	$\hat{\imath}$	i^v	1^v	$\hat{1}$	$\hat{\jmath}$	j^v
j^v	i^v	$\hat{\imath}$	$\hat{1}$	1^v	$\hat{\jmath}$	j^v

Quaternions: $i * i = -1$; Wave Numbers: $\hat{1} * \hat{1} = \hat{1}$
Quaternions: $i * -i = 1$; Wave Numbers: $\hat{1} * 1^v = 1^v$
Quaternions: $i * j = k$; Wave Numbers: $\hat{1} * \hat{\imath} = \hat{\jmath}$
Quaternions: $i * -j = -k$; Wave Numbers: $\hat{1} * i^v = j^v$
Quaternions: $i * k = -j$; Wave Numbers: $\hat{1} * \hat{\jmath} = i^v$
Quaternions: $i * -k = j$; Wave Numbers: $\hat{1} * j^v = \hat{\imath}$

The results between the quaternion and Wave Number multiplication tables largely correspond. The primary differences arise from the absence of a scalar multiplier or result in the Wave Number system. For instance, in quaternions, $i * i = -1$, whereas in the R3 system, $\hat{1} * \hat{1} = \hat{1}$. Similarly, multiplying $1^v * \hat{1} = \hat{1}$. In both cases, the Wave Number R3 multiplications leave the operand unchanged.

7.1.2 Universal Rules of Rotation

The Return Rule

The first rule of rotation in R3 is the Return Rule. It states that continuously rotating a point by the same angle will return the point to its original position when the total rotation is a multiple of 360^o.

The following examples show that R3 rotation meets the Return Rule:

$\hat{1} * \hat{\imath} = \hat{\jmath}, \hat{1} * \hat{\jmath} = i^v, \hat{1} * i^v = j^v, \hat{1} * j^v = \hat{\imath}$

$\hat{1} * (\hat{1} + \hat{\imath} + \hat{\jmath}) = (\hat{1} + \hat{\jmath} + i^v)$
$\hat{1} * (\hat{1} + \hat{\jmath} + i^v) = (\hat{1} + i^v + j^v)$
$\hat{1} * (\hat{1} + i^v + j^v) = (\hat{1} + j^v + \hat{\imath})$
$\hat{1} * (\hat{1} + j^v + \hat{\imath}) = (\hat{1} + \hat{\imath} + \hat{\jmath})$

$\frac{2\pi^\wedge}{3} \circlearrowleft^{(4^\wedge+5i^\wedge+6j^\wedge)} (1^\wedge + 2i^\wedge + 3j^\wedge) = (2.28958^\wedge + 1.52473i^\wedge + 2.53634j^\wedge)$
$\frac{2\pi^\wedge}{3} \circlearrowleft^{(4^\wedge+5i^\wedge+6j^\wedge)} (2.28958^\wedge + 1.52473i^\wedge + 2.53634j^\wedge) = (1.69743^\wedge + 2.70904i^\wedge + 1.94419j^\wedge)$
$\frac{2\pi^\wedge}{3} \circlearrowleft^{(4^\wedge+5i^\wedge+6j^\wedge)} (1.69743^\wedge + 2.70904i^\wedge + 1.94419j^\wedge) = (1^\wedge + 2i^\wedge + 3j^\wedge)$

Orthogonal Rule

The second rule of rotation in R3 is the Orthogonal Rule. It states that rotation by a unitary value is 90^o. Rotating by an Opposite Value with a $^\wedge$ sign indicates a counterclockwise direction, while rotating by an Opposite Value with a ^{v} sign indicates a clockwise direction. This rule is fully represented in the R3 multiplication table.

Reversal Rule

The third universal rule of rotation in R3 is the Reversal Rule. This rule states that a rotation can be reversed by multiplying the result of the initial rotation by the same operator, but with the sign reversed. In other words, a 90^o turn in one direction can be undone by a 90^o turn in the opposite direction. Even when a point is rotated around its own axis, resulting in no actual displacement, the Reversal Rule still applies.

The following examples show that R3 rotation meets the Reversal Rule:

$1^\wedge * i^\wedge = j^\wedge$ and $1^v * j^\wedge = i^\wedge$
$j^\wedge * i^v = 1^\wedge$ and $j^v * 1^\wedge = i^v$

$\frac{2\pi^\wedge}{3} \circlearrowleft^{(4^\wedge+5i^\wedge+6j^\wedge)} (1^\wedge + 2i^\wedge + 3j^\wedge) = (2.28958^\wedge + 1.52473i^\wedge + 2.53634j^\wedge)$ and
$\frac{2\pi^v}{3} \circlearrowleft^{(4^\wedge+5i^\wedge+6j^\wedge)} (2.28958^\wedge + 1.52473i^\wedge + 2.53634j^\wedge) = (1^\wedge + 2i^\wedge + 3j^\wedge)$

Remain Rule

The fourth universal rule of rotation in R3 is the Remain Rule. This rule states that a point remains in the same position when it is rotated about its own axis.

The following examples show that R3 rotation meets the Remain Rule:

$1^v * 1^\wedge = 1^\wedge$
$1^v * 1^v = 1^v$
$i^v * i^\wedge = i^\wedge$
$\frac{2\pi^\wedge}{3} \circlearrowleft^{(4^\wedge+5i^\wedge+6j^\wedge)} (4^\wedge + 5i^\wedge + 6j^\wedge) = (4^\wedge + 5i^\wedge + 6j^\wedge)$

Zero Rules

The final universal rule of rotation in R3 is the Zero Rules. The following examples show that R3 rotation meets each case:

Part(a): $\alpha * \beta + \alpha *^- \beta = 0$

$(1^v * i^v + 1^v * \hat{\imath}) = (\hat{\jmath} + j^v) = 0$
$(j^v * \hat{\imath} + j^v * i^v) = (\hat{1} + 1^v) = 0$

Part(b): $\alpha * \beta + \beta * \alpha = 0$ where α and β are not the same Opposite Type and Sign

$1^v * i^v + i^v * 1^v = \hat{\jmath} + j^v = 0$
$j^v * \hat{\imath} + \hat{\imath} * j^v = \hat{1} + 1^v = 0$

Part(c): $\alpha + \beta(\beta * \alpha) = 0$ where α and β are not the same Opposite Type and Sign.

$1^v + j^v(j^v * 1^v) = 1^v + j^v * \hat{\imath} = 1^v + \hat{1} = 0$
$\hat{\imath} + 1^v(1^v * \hat{\imath}) = \hat{\imath} + 1^v * j^v = \hat{\imath} + i^v = 0$

7.1.3 Comparing Wave Number and Quaternion Efficiency

This comparison of Wave Numbers and quaternions evaluates their efficiency based on the rotation operation due to its computational complexity. Computer games and other high quality graphics frequently use rotation operations to implement the physics of movement, making this an important consideration for performance[14].

Unit of Rotation

The magnitude and unit rotation of a point (x, y, z) are determined by the following calculations:

- $R = \sqrt{(|x|^2 + |y|^2 + |z|^2)}$
- Unit of Rotation $= (x/R, y/R, z/R)$
- Therefore, the calculation requires 3 multiplications, 1 square root and 3 divisions.

Wave Number Processor Workload

The formula for Wave Numbers rotation is:

$$v_{rot} = (u_r \cdot v)u_r + sin\theta(u_r \times v) + cos\theta((u_r \times v) \times u_r)$$

Wave Numbers have 3 opposite types, leading to multiplications occurring in sets of 3. This results in the following calculations during rotation:

- Unit of Rotation.
  - 3 multiplications, 1 square root and 3 divisions.
- vrot.
  - Element 1 - $(u_r \cdot u_r)v_r$.
    - 3 multiplications and 2 additions to get dot product.
    - 3 multiplications of dot product and u_r.
  - Element 2 - $sin\theta(u_r \times v)$.
    - 6 multiplications and 3 additions to get cross product.
    - 3 multiplications of cross product with sine and 1 sine calculation.
  - Element 3 - $cos\theta((u_r \times v) \times u_r)$.

- First cross product already calculated for Element 2.
- 6 multiplications and 3 additions to get second cross product.
- 3 multiplications of cross product with cosine and 1 cosine calculation.

- Total: 27 multiplications, 8 additions, 1 square root, 3 divisions, 1 sine and 1 cosine for a single rotation.

Quaternion Processor Workload

Quaternions consist of 4 different types, resulting in multiplications occurring in sets of 4.

- Unit of Rotation
  - 3 multiplications, 1 square root and 3 divisions.
- Quaternion of Rotation.
  - The formula for the quaternion of Rotation is $Qr = cos(\frac{\theta}{2}) + (x_i + y_j + z_k) * sin(\frac{\theta}{2})$.
  - 3 multiplications, 1 division, 1 cosine and 1 sine.
- $Qr * Qv$
  - 16 multiplications of Qr and Qv.
  - 12 additions to consolidate result into scalar, i, j and k quaternion buckets.
- Subtotal per multiplication
  - 22 multiplications, 12 additions, 1 square root, 4 divisions, 1 sine and 1 cosine.
- Conversion of quaternion to R3
  - If there is more than 1 rotation in a chain, then the overall rotation must be calculated starting in reverse. i.e. if the Rotations Q1 was followed by Q2 and then Q3 the reverse rotation is (Q3*Q2)*Q1).
    - 16$*$ (number of rotations - 1) multiplications.
    - 12$*$ (number of rotations - 1) additions.
    - In the case of 3 rotations this gives $16 * 2 = 32$ multiplications and 32 additions.
  - 16 multiplications with conjugate of overall reverse rotation followed by 12 additions.
- Total
  - 38 multiplications, 24 additions, 1 square root, 4 divisions, 1 sine and 1 cosine for a single rotation.

Efficiency Comparison

- 1 Multiplication
  - Wave Numbers - 27 multiplications, 8 additions, 1 square roots, 3 divisions, 1 sine and 1 cosine.
  - Quaternions - 38 multiplications, 24 additions, 1 square roots, 4 divisions, 1 sine and 1 cosine.
- 2 Multiplications
  - Wave Numbers - 54 multiplications, 16 additions, 2 square roots, 6 divisions, 2 sine and 2 cosine.
  - Quaternions - 76 multiplications, 48 additions, 2 square roots, 8 divisions, 2 sine and 2 cosine.

- 3 Multiplications
  - Wave Numbers - 81 multiplications, 24 additions, 3 square roots, 9 divisions, 3 sine and 3 cosine.
  - Quaternions - 114 multiplications, 72 additions, 3 square roots, 12 divisions, , 3 sine and 3 cosine.
- 4 Multiplications
  - Wave Numbers - 108 multiplications, 32 additions, 4 square roots, 12 divisions, 4 sine and 4 cosine.
  - Quaternions - 152 multiplications, 96 additions, 4 square roots, 16 divisions, 4 sine and 4 cosine.

Conclusion

This comparison of Wave Number and quaternion multiplication demonstrates that Wave Numbers are more efficient. Additionally, Wave Numbers provide a clearer representation as they are always situated in 3D space

7.2 Multiplication

7.2.1 Multiplication Links to Rotation and Rotication

Rotication Formula

R3 rotication is calculated using the magnitude of the axis, $\sqrt{(|x_a|^2 + |y_a|^2 + |z_a|^2)}$ and the Wave Number formula as follows:

$$v_{Rotication} = \sqrt{(|x_a|^2 + |y_a|^2 + |z_a|^2)} * ((u_r \cdot v)u_r + sin\theta(u_r \times v) + cos\theta((u_r \times v) \times u_r))$$

The formula for multiplication was defined earlier as below where $_a$ denotes the operator or axis, $_o$ represents the operand, and x, y, and z are the coordinates.
$\hat{}/^v$ value: $(x_a * x_o + y_a * z_o + z_a * y_o)$
$\hat{i}/i^v$ value: $(y_a * y_o + x_a * z_o + z_a * x_o)$
$\hat{j}/j^v$ value: $(z_a * z_o + x_a * y_o + y_a * x_o)$

Take the case of the multiplication of the points at (x_a, y_a, z_a) and (x_o, y_o, z_o). This can be broken down into:
$(|x_a| * 1^{\ddagger x_a}) * (x_o + y_o + z_o) + (|y_a| * i^{\ddagger y_a})(x_o + y_o + z_o) + (|z_a| * j^{\ddagger z_a}) * (x_o + y_o + z_o)$

Multiplication is defined as rotation by a unitary. Therefore:
$1^{\ddagger x_a} * (x_o + y_o + z_o) = \frac{\pi}{2}^{\ddagger x_a} \circlearrowleft^x (x_o + y_o + z_o)$ and
$(|x_a| * 1^{\ddagger x_a}) * (x_o + y_o + z_o) = \frac{\pi}{2}^{\ddagger x_a} R \circlearrowleft^x (x_o + y_o + z_o)$

In the case where the angle of rotation is $\frac{\pi}{2}^{\ddagger x_a}$ and the axis is represented by $(x_a, 0, 0)$, the vector v by (x_o, y_o, z_o), the unit of rotation is $(1^{\ddagger x_a}, 0, 0)$, the formula of rotication can be written as:

$v_{Rotication} =$

$\sqrt{(|x_a|^2+|0^2|+|0^2|)} * (((1^{\ddagger x_a},0,0)\cdot v)(1^{\ddagger x_a},0,0)+sin\frac{\pi}{2}((1^{\ddagger x_a},0,0)\times v)+cos\frac{\pi}{2}((u_r \times v)\times u_r))$
$= |x_a| * ({}^{?}|x_o|(1^{\ddagger x_a},0,0)+1(1^{\ddagger x_a} * z_o + 1^{\ddagger x_a} * y_o) + 0 * ((u_r \times v)\times u_r))$
$= |x_a| * (x_o^{\ddagger x_a} + 1^{\ddagger x_a} * z_o + 1^{\ddagger x_a} * y_o)$
$= (x_a * x_o + x_a * z_o + x_a * y_o) EquationA$

Note that ${}^{?}|x_o|1^{\ddagger x_a} = x_o$. Take 4 possible scenarios for values of x_a and x_o:
1. x_a and x_o both have $\hat{}$ Opposite signs. e.g. : $x_a = \hat{5}$ and $x_o = \hat{4}$. Therefore, $((1^{\ddagger x_a},0,0)\cdot v)(1^{\ddagger x_a},0,0) = ((\hat{1},0,0)\cdot(\hat{4},0,0))(\hat{1},0,0) = 4(\hat{1},0,0) = \hat{4}$

2. x_a has a $\hat{}$ Opposite Sign and x_o has a ^{v} Opposite signs. e.g. : $x_a = \hat{5}$ and $x_o = 4^v$. Therefore, $((1^{\ddagger x_a},0,0)\cdot v)(1^{\ddagger x_a},0,0) = ((\hat{1},0,0)\cdot(4^v,0,0))(\hat{1},0,0) =^- 4(\hat{1},0,0) = 4^v$

3. x_a has a ^{v} Opposite Sign and x_o has a $\hat{}$ Opposite signs. e.g. : $x_a = 5^v$ and $x_o = \hat{4}$. Therefore, $((1^{\ddagger x_a},0,0)\cdot v)(1^{\ddagger x_a},0,0) = ((1^v,0,0)\cdot(\hat{4},0,0))(1^v,0,0) =^- 4(1^v,0,0) = \hat{4}$

4. x_a has a ^{v} Opposite Sign and x_o has a ^{v} Opposite signs. e.g. : $x_a = 5^v$ and $x_o = 4^v$. Therefore, $((1^{\ddagger x_a},0,0)\cdot v)(1^{\ddagger x_a},0,0) = ((1^v,0,0)\cdot(4^v,0,0))(1^v,0,0) = 4(1^v,0,0) = 4^v$

It can be seen that in each of these cases the result is x_o due to the workings of the formula. Note also that $|x_a| * 1^{\ddagger x_a} = x_a$.

Similarly, in the case where the angle of rotation is $\frac{\pi}{2}^{\ddagger y_a}$ and the axis is represented by $(0, y_a, 0)$, the vector v by (x_o, y_o, z_o), the unit of rotation is $(0, i^{\ddagger y_a}, 0)$, the formula of rotication results in:

$$(y_a * y_o + y_a * z_o + y_a * x_o) EquationB$$

Finally, when the angle of rotation is $\frac{\pi}{2}^{\ddagger z_a}$ and the axis is represented by $(0, 0, z_a)$, the vector v by (x_o, y_o, z_o), the unit of rotation is $(0, 0, j^{\ddagger z_a})$, the formula of rotication results in:

$$(z_a * z_o + z_a * y_o + z_a * x_o) EquationC$$

The sum of equations A, B and C can be rearranged to match the formula for multiplication:
$\hat{}/^v$ value: $(x_a * x_o + y_a * z_o + z_a * y_o)$
$\hat{i}/i^v$ value: $(y_a * y_o + x_a * z_o + z_a * x_o)$
$\hat{j}/j^v$ value: $(z_a * z_o + x_a * y_o + y_a * x_o)$

In conclusion, this proves that multiplication is the sum of rotications of $\frac{\pi}{2}$ around the x, y and z axes.

7.2.2 Comparing Wave Number and Quaternion Squares

Quaternions

Quaternions are expressed in the form $(a+bi+cj+dk)$. A quaternion with only a value for a is a scalar, whereas quaternions with $a = 0$ are vectors or pure quaternions. They represent vectors in real $\mathbb{R}3$ space. For example, $(2i + 3j + 4k)$ is a pure quaternion, and -29 is a scalar.

In quaternion algebra, squaring a pure quaternion results in a scalar. For example, $(2i - 3j + 4k)(2i - 3j + 4k) = -29$. This result directly follows from the rules of quaternion multiplication.

Wave Numbers

Given the generic expression $(a + bi + cj)$:

$$(a + bi + cj)^2 = a^2 + a * bi + a * cj + bi * cj + bi * a + bi^2 + cj * bi + cj^2 + cj^a$$

Note that as a result of the Zero Rules of Rotation $a * b + b * a = 0$, therefore
$(a + bi + cj)^2 = (a^2 + bi^2 + cj^2)$

This demonstrates that squaring an expression results in the squares of the individual Opposite Values. Consequently, the square root of an expression is the square root of the individual Opposite Values. This formula is surprisingly easy to use.

In Wave Numbers, the equivalent operation to squaring a quaternion $(2i-3j+4k)$ is $(\hat{2} + 3i^v + 4\hat{\jmath}) * (\hat{2} + 3i^v + 4\hat{\jmath})$. This yields $(\hat{4} + 9\hat{\imath} + 16\hat{\jmath})$.

Intuitively, the Wave Number result seems more realistic, as it provides a point of greater magnitude in 3D space. In contrast, quaternions produce a scalar, which does not exist in 3D space.

7.2.3 Conjugates

Roots

The need for conjugates in classical mathematics can be traced back to square roots. In $\mathbb{R}2$ more than one root is available. When dealing with roots, especially in fractions, multiplying by the conjugate above and below the divisor line helps remove the root from the divisor, simplifying the expression. For example:

$$\frac{1}{(a + \sqrt{b})} * \frac{(a - \sqrt{b})}{(a - \sqrt{b})} = \frac{(a - \sqrt{b})}{(a^2 - b)}$$

There is no need for a conjugate in Wave Numbers R3 as multiplying above and below the divisor line with the divisor, in cases like the above example, eliminates the root from the divisor.

$$\frac{(\hat{1})}{(a+\sqrt{b})} * \frac{(a+\sqrt{b})}{(a+\sqrt{b})} = \frac{\hat{1}(a+\sqrt{b})}{(a^2+b)}$$

Quaternion Conjugate

As demonstrated in the section on rotation, quaternions require the quaternion conjugate to complete the quaternion sandwich, which transforms the quaternion back into 3D space. In contrast, R3 Wave Numbers are always in 3D space and do not require this mechanism.

Eliminating Complex Numbers

Conjugates are used in a general fashion to simplify complex numbers. For example:

$$\frac{(a+bi)}{(c+di)} * \frac{(c-di)}{(c-di)} = \frac{(ac+bd+(bc-ad)i}{(c^2+d^2)}$$

In contrast, R3 Wave Numbers are real numbers and do not require this mechanism.

7.2.4 Cardano

R3 Wave Numbers fully implement Cardano's (1501-1576) alternative rule of signs, where he proposed that multiplying a minus by a minus gives a minus. [2] Cardano viewed positives and negatives as two distinct areas that should be kept separate. According to his rule, multiplying positives results in a positive, while multiplying negatives results in a negative.

7.3 Division

7.3.1 Self Division

In quaternions, dividing a quaternion by itself results in the scalar 1. However, in Wave Numbers R3, division of an expression by itself does not always yield a unitary result. The result of dividing an operand by an operator is another expression. Multiplying the operator by this result returns the original operand. For example:

$$(\hat{2}+3i^v+4\hat{\jmath})/(\hat{2}+3i^v+4\hat{\jmath}) = (0.56\hat{9}+1.033i^v+1.089\hat{\jmath})$$

This result can be verified by multiplying it by the original operator:

$$(\hat{2}+3i^v+4\hat{\jmath}) * (0.56\hat{9}+1.033i^v+1.089\hat{\jmath}) = (\hat{2}+3i^v+4\hat{\jmath})$$

Division in Wave Numbers is the inverse of multiplication, meaning it reverses both scalar multiplication and a rotation. Multiplication in Wave Numbers moves a point from one location to another, while division moves a point back, effectively reversing scalar multiplication and circular motion.

In R3, self-division results in a unitary value when the division is by the same, single Opposite Value. For example:

$\hat{1} * \hat{1} = \hat{1} \implies \hat{1}/\hat{1} = \hat{1}$
$\hat{7} * \hat{1} = \hat{7} \implies \hat{7}/\hat{7} = \hat{1}$
$9^v * \hat{1} = \hat{9} \implies \hat{9}/9^v = \hat{1}$

Moreover, division can never result in the Counter 1 as it cannot be the location of a point in R3.

Some answers to other types of division can consist of unitaries. For example:
$(\hat{4} + 5\hat{\imath} + 6\hat{\jmath}) * (\hat{1}) = (\hat{4} + 6\hat{\imath} + 5j^v) \implies (\hat{4} + 6\hat{\imath} + 5j^v)/(\hat{4} + 5\hat{\imath} + 6\hat{\jmath}) = \hat{1}$
$(\hat{4} + 5\hat{\imath} + 6\hat{\jmath})({}^v + \hat{\imath}) = (10^v + i^v + 9\hat{\jmath}) \implies (10^v + i^v + 9\hat{\jmath})/(\hat{4} + 5\hat{\imath} + 6\hat{\jmath}) = (1^v + \hat{\imath})$
$(\hat{4} + 5\hat{\imath} + 6\hat{\jmath})(\hat{} + \hat{\imath} + \hat{\jmath}) = (\hat{3} + 7\hat{\imath} + 5\hat{\jmath}) \implies (\hat{3} + 7\hat{\imath} + 5\hat{\jmath})/(\hat{4} + 5\hat{\imath} + 6\hat{\jmath}) = (\hat{1} + \hat{\imath} + \hat{\jmath})$

7.3.2 Ratios

In Cecilia Hamm's "Making Sense of Negative Numbers," she states, "Arnauld (1612-1694) claimed that the basic principle of multiplication is that the ratio of unity to one factor is equal to the ratio of the second factor. i.e., given the product a x b, then $1/a = b/(a \text{ x } b)$ or $1/b = a/(a \text{ x } b)$." This is known as the axiom of ratios.

Classical mathematics sign rules state that multiplying two negatives yields a positive. Arnauld believed that this rule violated the principle that "the ratio of unity to one factor is equal to the ratio of the second factor." For example: Given $a = -4$ and $b = -5$:

For $1/a$, it can be expressed as $1 : -4 = -5 : 20$. Here, ":" represents the ratio, which is the relative size of two numbers. This illustrates the problem: the ratio of the larger number 1 to the smaller number -4 is the same as the ratio of the smaller number -5 to the larger number 20. In other words, "a larger number is to a smaller number (1 to -4) as a smaller number is to a larger number (-5 to 20)."

For $1/b$, it can be expressed as $1 : -5 = -4 : 20$. Again, this shows the problem that the ratio of the larger number 1 to the smaller number -5 is the same as the ratio of the smaller number -4 to the larger number 20.

However, in Wave Numbers, only the comparison of Opposite Values' magnitudes is possible. Wave Numbers use the absolute values of Opposite Values for this purpose, as follows.

Given $a = 4i^v$ and $b = 5j^v$ and $ab = 2\hat{0}$:

For $\hat{1}/a$, it can be expressed as $|\hat{1}| : |4i^v| = |5j^v| : |20^v| \implies 1 : 4 = 5 : 20$. Both ratios are of a smaller Counter to a bigger Counter. Therefore a breach of the basic principle of multiplication does not happen. For $\hat{1}/b$, it can be expressed as $|\hat{1}| : |5j^v| = |4i^v| : |20^v| \implies 1 : 5 = 4 : 20$. Both ratios are of a smaller Counter to a bigger Counter. Therefore a breach of the basic principle of multiplication does not happen.

7.4 Other

7.4.1 Order

In Cecilia Hamm's "Making Sense of Negative Numbers," she states:

A: "For natural numbers, if $a > b$ and $\frac{a}{b} = \frac{c}{d}$ then $c > d$ and there exists a number $n > 0$ such that $a = nc$ and $b = nd$. Hence for any $a, b, c \in \mathrm{N}$ if $a > b$ then $ac > bc$. Including zero in the domain and taking $c = 0$ will lead to a contradiction: if $a > b$ then $0 > 0$."

However, in Wave Numbers, if $a = 0$, then the statement $a > b$ cannot be true, as no Opposite Value is less than 0. Thus, this is not a valid example in Wave Numbers. Additionally, division by 0 is undefined in Wave Numbers.

The statement if $a > b$ and $\frac{a}{b} = \frac{c}{d}$ then $c > d$ needs to be qualified in Wave Numbers to exclude the cases where $b = 0$ or $d = 0$.

B: "Including negative numbers in the domain and taking $a \neq 0$ and $b = -a$ implies that if $a > -a$ then a$c < -ac$. Taking $c = -1$ will lead to a contradiction: if $a > -a$ then $a(-1) < -a(-1)$, in short: if $a > -a$ then $-a < a$. This contradicts the basic property of order."

In Wave Numbers, only the magnitudes of Opposite Values can be compared using the concept of "less than." This means only their Counters can be compared. The absolute value symbol is used to refer specifically to an Opposite Value's Counter. For example, $|4| = |4^v|$ and $|a| = |^-a|$. Therefore, the statement $|a| > |^-a|$ can never be true. This ensures that no contradiction of the basic property of order can occur.

8 Conclusions and Further Development

Both the theory and examples presented in this paper demonstrate that the Wave Number system offers a viable 3D algebra. Analysis suggests that this algebra is more efficient than quaternions, potentially leading to significant savings in running software that involves 3D movement.

Future work will include the formal presentation of R1 and R2 algebras. The operations rotvision and roticvision will be formally defined. Additionally, the application of Wave Numbers to quantum programming is an area of interest for further exploration.

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